Topology in condensed matter physics
Exercise sheet 8

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8.1 Spectrum of Kitaev's Majorana chain Hamiltonian

Find the single-particle spectrum of the periodic Kitaev Majorana chain

$$\mathcal{H} = \sum_{j=1}^{N} \mu \left(c_j^\dagger c_j - 1/2 \right) + w c_j^\dagger c_{j+1} + w c_{j+1}^\dagger c_j + \Delta c_{j+1}^\dagger c_j^\dagger + \Delta^* c_j c_{j+1}, \tag{1}$$

with $c_{N+1} \equiv c_1$. One route is the following

a. (1 point) Use the spinor $\tilde{c}_j = (c_j, c_j^\dagger)$ to reexpress the Hamiltonian in matrix form.

Solution:

$$\mathcal{H} = \sum_{j=1}^{N} \mu \left(c_j^\dagger c_j - 1/2 \right) + w c_j^\dagger c_{j+1} + w c_{j+1}^\dagger c_j + \Delta c_{j+1}^\dagger c_j^\dagger + \Delta^* c_j c_{j+1} \tag{2}$$

$$\Big| \quad c_j c_j^\dagger = 1 - c_j^\dagger c_j, \quad c_j^\dagger c_{j+1} = -c_{j+1} c_j^\dagger \tag{3}$$

$$= \sum_{j=1}^{N} \mu \left(c_j^\dagger c_j - c_j c_j^\dagger \right) + w \left(c_j^\dagger c_{j+1} - c_{j+1} c_j^\dagger \right) + \Delta c_{j+1}^\dagger c_j^\dagger + \Delta^* c_j c_{j+1} \tag{4}$$

$$= \sum_{j=1}^{N} \begin{pmatrix} c_j^\dagger & c_j \end{pmatrix} \overbrace{\begin{pmatrix} \frac{\mu}{2} & 0 \\ 0 & \frac{-\mu}{2} \end{pmatrix}}^{=:A_1} \begin{pmatrix} c_j \\ c_j^\dagger \end{pmatrix} + \begin{pmatrix} c_{j+1}^\dagger & c_{j+1} \end{pmatrix} \overbrace{\begin{pmatrix} 0 & \Delta \\ 0 & -w \end{pmatrix}}^{=:A_2} \begin{pmatrix} c_j \\ c_j^\dagger \end{pmatrix} \tag{5}$$

$$+ \begin{pmatrix} c_j^\dagger & c_j \end{pmatrix} \overbrace{\begin{pmatrix} w & 0 \\ \Delta^* & 0 \end{pmatrix}}^{=:A_3} \begin{pmatrix} c_{j+1} \\ c_{j+1}^\dagger \end{pmatrix} \tag{6}$$

$$\Big| \quad \tilde{c}_j := (c_j, \; c_j^\dagger) \tag{7}$$

$$= \underline{\underline{\sum_{j=1}^{N} \tilde{c}_j^\dagger A_1 \tilde{c}_j + \tilde{c}_{j+1}^\dagger A_2 \tilde{c}_j + \tilde{c}_j^\dagger A_3 \tilde{c}_{j+1}}} \tag{8}$$

b. (5 points) Conduct a Fourier transform by introducing the spinor $d_j = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{2\pi \mathrm{i}/N jk} \tilde{c}_j$ and conducting the sum over the real space index j. Hint: The sums are geometric series, and $\sum_{j=1}^{N} e^{\mathrm{i}ja} = N\delta_{0,a}$.

Solution:

$$H = \sum_{j=1}^{N} \tilde{c}_j^\dagger A_1 \tilde{c}_j + \tilde{c}_{j+1}^\dagger A_2 \tilde{c}_j + \tilde{c}_j^\dagger A_3 \tilde{c}_{j+1}$$

$$\left|\ \tilde{c}_j = \frac{1}{\sqrt{N}} \sum_k e^{ikj}\ \tilde{c}_k, \quad c_j^\dagger = \frac{1}{\sqrt{N}} \sum_k e^{-ikj}\ \tilde{c}_k^\dagger, \quad k = \frac{2\pi m}{N}, \text{ with } m \in \{0, 1, ..., N-1\}\right.$$

$$\left|\ \tilde{c}_{j+1} = \frac{1}{\sqrt{N}} \sum_k e^{ikj} e^{ik} \tilde{c}_k, \quad \tilde{c}_{j+1}^\dagger = \frac{1}{\sqrt{N}} \sum_k e^{-ikj} e^{-ik} \tilde{c}_k^\dagger\right.$$

$$= \frac{1}{N} \sum_{k,k'} \left\{ \sum_{j=1}^{N} e^{i(-k+k')j} \left[\tilde{c}_k^\dagger A_1 \tilde{c}_{k'} + e^{-ik} \tilde{c}_k^\dagger A_2 \tilde{c}_{k'} + e^{ik'} \tilde{c}_k^\dagger A_3 \tilde{c}_{k'} \right] \right\}$$

$$\left|\ \sum_{j=1}^{N} e^{i(-k+k')j} = N \delta_{(-k+k'),0}\right.$$

$$= \sum_k \tilde{c_k}^\dagger \left[A_1 + e^{-ik} A_2 + e^{ik} A_3 \right] \tilde{c}_k$$

$$= \sum_k \tilde{c_k}^\dagger \begin{pmatrix} \frac{\mu}{2} + w e^{ik} & \Delta e^{ik} \\ \Delta^* e^{ik} & -\frac{\mu}{2} - w e^{-ik} \end{pmatrix} \tilde{c}_k$$

Note: This description considers particles and holes at the same k. For k and $-k$, use the Nambu spinor $\tilde{c}_k = (c_k, c_{-k}^\dagger)$.

8.2 Reformulate a general fermionic Hamiltonian in Majorana form

Consider a general discrete noninteracting fermionic Hamiltonian

$$\mathcal{H} = \sum_{l,m=1}^{N} c_l^\dagger A_{l,m} c_m + c_l^\dagger B_{l,m} c_m^\dagger \ . \tag{9}$$

a. (3 points) Reformulate Eq. (9) in Majorana form

$$\mathcal{H} = \frac{\mathrm{i}}{2} \sum_{l,m=1}^{2N} \gamma_l D_{l,m} \gamma_m. \tag{10}$$

Here, $\gamma_{2j} = \frac{1}{\sqrt{2}} \left(c_j + c_j^\dagger \right)$ and $\gamma_{2j+1} = \frac{1}{\sqrt{2}\mathrm{i}} \left(c_j - c_j^\dagger \right)$, for integers j. Show that the matrix D can be chosen real and skew-symmetric, i.e., $D^T = -D$.
Hint: *Use matrix notation*

Solution: With $c_j^\dagger = \frac{1}{\sqrt{2}}\Big[\gamma_j - i\gamma_{j+1}\Big]$, we obtain

$$H = \frac{1}{2}\sum_{l,m=1}^{N}\Big[\gamma_l - i\gamma_{l+1}\Big]\underbrace{\Big\{A_{lm}\Big[\gamma_m + i\gamma_{m+1}\Big] + B_{lm}\Big[\gamma_m - i\gamma_{m+1}\Big]\Big\}}_{\Big[A_{lm}+B_{lm}\Big]\gamma_m + \Big[iA_{lm}-B_{lm}\Big]\gamma_{m+1}} + \text{h.c.}$$

$$\underline{\underline{= \frac{1}{2}\sum_{l,m=1}^{N}\Big[A_{lm}+B_{lm}\Big]\gamma_l\gamma_m + \Big[A_{lm}-B_{lm}\Big]\gamma_{l+1}\gamma_{m+1}}}$$

Note, that we could also introduce a superindex. For the constraint of hermiticity, we can choose $A, B \in \mathbb{R}$ and $A^T = -A, \quad B^T = -B$

$$H^\dagger = \frac{1}{2}\sum_{l,m=1}^{N}\Big[A_{lm}+B_{lm}\Big]^\dagger\gamma_m^\dagger\gamma_l^\dagger + \Big[A_{lm}-B_{lm}\Big]^\dagger\gamma_{m+1}^\dagger\gamma_{l+1}^\dagger$$

$$\Bigg|\quad \gamma_m\gamma_l = -\gamma_l\gamma_m, \quad A, B \in \mathbb{R}, \quad A^T = -A, \quad B^T = -B$$

$$\underline{\underline{= H}}\;.$$

End