Topology in condensed matter physics
Exercise sheet 7

University of Hamburg
Thore Posske

7.1 Conventional time reversal operator and time reversal symmetry

This exercise shall provide some physical intuition on time reversal and time reversal symmetry. Imagine a system of spin-1/2 particles, the annihilation operators of which are given by $\Psi_\uparrow(x)$ and $\Psi_\downarrow(x)$, where x is a vector in real space. The "conventional" time reversal operator on this system is defined to be an antiunitary operator T with

$$\begin{aligned} T\Psi_\uparrow(x) &= \Psi_\downarrow(x)T, \\ T\Psi_\downarrow(x) &= -\Psi_\uparrow(x)T. \end{aligned} \tag{1}$$

Formulated with the two-component spinor $\Psi(x) = (\Psi_\uparrow(x), \Psi_\downarrow(x))$, this is written as $T\Psi(x) = \mathrm{i}\sigma_y\Psi(x)T$, where σ_y is the second Pauli matrix, and hence, in the single particle picture, T can be written as $T = \mathrm{i}\sigma_y\mathcal{K}$, where $\mathcal{K}$ denotes complex conjugation. To clarify further nomenclature: the time conjugation A' of an operator A is given by $A' = TAT^{-1}$.

Note: The Pauli matrices are $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, *and* $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *These matrices fulfill the algebra* $\sigma_i\sigma_j = \delta_{i,j}\underline{\underline{1}} + \mathrm{i}\sum_k \epsilon_{i,j,k}\sigma_k$, *where* $\underline{\underline{1}}$ *is the* 2×2 *identity matrix and* ϵ *the Levi-Civita symbol.*

1. (1 point) Show that the time evolution operator $U(t) = exp(-\mathrm{i}/\hbar Ht)$ obeys $TU(t)T^{-1} = U(-t)$ if the Hamiltonian H is time reversal symmetric. I.e., the time conjugation of $U(t)$ is $U(-t)$.

Solution:

$$\begin{aligned} TU(t)T^{-1} &= T\exp\left\{-\frac{-i}{\hbar}Ht\right\}T^{-1} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}T\left(\frac{-i}{\hbar}\right)^n H^n t^n T^{-1} \\ &\quad \Big| \quad H' = THT^{-1}, \quad H' = H, \quad TH^nT^{-1} = \left(THT^{-1}\right)^n \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-i}{\hbar}\right)^n H^n(-t)^n \\ &\underline{\underline{= U(-t)}} \end{aligned}$$

2. (1 point) In momentum space, the annihilation operators are $c_{k,\sigma} = \int_{\mathbb{R}} dx \frac{e^{\mathrm{i}kx}}{\sqrt{2\pi}}\Psi_\sigma(x)$. How does time reversal symmetry act on $c_{k,\sigma}$?

Solution:

$$\begin{aligned}
Tc_{k\sigma}T^{-1} &= T\Big(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\Psi_\sigma(x)e^{ikx}dx\Big)T^{-1} \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\sum_{n=0}^{\infty}\frac{1}{n!}T\Big(\Psi_\sigma(x)(ixk)^n\Big)T^{-1}dx \\
&\quad \Bigg| \quad T\Psi_\sigma(x)T^{-1} = \zeta_\pm\Psi_{\tilde\sigma}(x), \quad \zeta_\pm = \begin{cases} 1 \text{ if } \sigma = \uparrow \\ -1 \text{ if } \sigma = \downarrow \end{cases}, \quad \sigma \neq \tilde\sigma \\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\sum_{n=0}^{\infty}\frac{(ix)^n}{n!}\zeta_\pm\Psi_{\tilde\sigma}(x)(-k)^n dx \\
&= \frac{\zeta_\pm}{\sqrt{2\pi}}\int_{\mathbb{R}}\Psi_{\tilde\sigma}(x)e^{-ikx}dx \\
&\underline{\underline{= \zeta_\pm c_{-k\tilde\sigma}}}
\end{aligned}$$

3. (3 points) What is the time reversal conjugation of the spin operators $J^\lambda(x) = \frac{1}{2}\Psi^\dagger(x)\sigma_\lambda\Psi(x)$ for the spin in x-, y-, and z-direction, respectively. Use matrix notation to simplify your calculations.

Solution:

$$\begin{aligned}
TJ^\lambda(x)T^{-1} &= \frac{1}{2}T\Psi^\dagger(x)\sigma_\lambda\Psi(x)T^{-1} \\
&= \frac{1}{2}\Psi^\dagger(x)\sigma_y\sigma_\lambda^*\sigma_y\Psi(x) \\
&\quad \Bigg| \quad \sigma_y\sigma_\lambda^*\sigma_y = \begin{cases} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\sigma_x \\ -\mathbb{I}_{2\times 2}\sigma_y = -\sigma_y \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\sigma_z \end{cases} \\
&= -\frac{1}{2}\Psi^\dagger(x)\sigma_\lambda\Psi(x) \\
&\underline{\underline{= \ -J^\lambda(x)}}
\end{aligned}$$

4. (1 point) Assume the single-particle Hamiltonian of spinful particles in a magnetic field $B(x) = \{B^x(x), B^y(x), B^z(x)\}$,

$$H = \int_{\mathbb{R}} dx \ \Psi^\dagger(x)v\left(-\mathrm{i}\partial_x\right)\Psi(x) + \sum_{\lambda\in\{x,y,z\}} B^\lambda(x)J^\lambda(x), \tag{2}$$

In general, here, the dispersion relation v can be any polynomial in its argument. Putting together what was found above, show that H is (conventionally) time reversal symmetric if and only if $B = 0$ (vanishing magnetic field) and v is an even polynomial.

Solution:

$$THT^{-1} = \int_{\mathbb{R}} T\Psi^\dagger(x)\ v\left(-\mathrm{i}\partial_x\right)\ \Psi(x)T^{-1}dx + \sum_{\lambda\in\{x,y,z\}} TB^\lambda(x)J^\lambda(x)T^{-1}$$

$$\Bigg|\quad T = i\sigma_y\mathcal{K}, \quad v(-i\partial_x) = v(\hat{p})$$

$$\Bigg|\quad Tv(\hat{p})T^{-1} = \frac{1}{\sqrt{2\pi}}\int v(x)e^{-i(-\hat{p})x}dx = v(-\hat{p})$$

$$= \int_{\mathbb{R}} \Psi^\dagger(x)\ v(-\hat{p})\ \Psi(x)dx \quad - \sum_{\lambda\in\{x,y,z\}} TB^\lambda(x)J^\lambda(x)$$

For a time-symmetric Hamiltonian $THT^{-1} = H$, the dispersion relation needs to be an even polynomial and the magnetic field needs to be zero $B = 0$.

5. (1 point) Is the square of the conventional time symmetry operator positive or negative? Do we have Kramers' partners?

Solution: In general, the square is $T^2 = \pm 1$. For a spin $\frac{1}{2}$-system it holds that $T^2 = -1$. For $H = THT^{-1}$, the system has Kramers' partners.

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