Topology in condensed matter physics
Exercise sheet 6

University of Hamburg
Thore Posske

6.1 10-fold way

Given a Hamiltonian $\mathcal{H}$ that has no unitary symmetry, a time reversal operator $\mathcal{T}$ is an antiunitary operator that commutes with $\mathcal{H}$, a particle-hole operator $\mathcal{P}$ is an antiunitary operator that anticommutes with $\mathcal{H}$ and a chiral symmetry $\mathcal{C}$ is a unitary operator that anticommutes with $\mathcal{H}$. These operators, if they exist, square to $\mathcal{T}^2, \mathcal{P}^2 \in \{-1, 1\}$, and $\mathcal{C}^2 = 1$.

Symmetries

$$T : \text{Antiunitary} \qquad [T, H]_- = 0 \tag{1}$$

$$P : \text{Antiunitary} \qquad [P, H]_+ = 0 \tag{2}$$

$$C : \text{Unitary} \qquad [C, H]_+ = 0 \tag{3}$$

Helpful properties

$$[A, BC]_\pm = [A, B]_- C + B[A, C]_\pm = [A, B]_\pm C \mp B[A, C]_-, \tag{4}$$

with $[A, B]_\pm = A \pm B$.

1. (2 points) Show that $\mathcal{C} = \mathcal{T}\mathcal{P}$, $\mathcal{T} = \mathcal{C}\mathcal{P}$, $\mathcal{P} = \mathcal{C}\mathcal{T}$ (understood up to possible scalar phases).

Solution:

$$C = TP : \qquad [H, TP]_+ = \underbrace{[H, T]_-}_{=0} P + T \underbrace{[P, H]_+}_{=0} = 0$$

$$T = CP : \qquad [H, CP]_- = \underbrace{[H, C]_+}_{=0} P - C \underbrace{[H, P]_+}_{=0} = 0$$

$$P = CT : \qquad [H, CT]_+ = \underbrace{[H, C]_+}_{=0} T + C \underbrace{[T, H]_-}_{=0} = 0$$

Kramer's degeneracy

Consider a Hamiltonian H with a time reversal symmetry T that squares to -1, i.e., an antiunitary symmetry T with $[H, T] = 0$ and $T^2 = -1$.

1. (2 points) Show that for each eigenvector $|\phi\rangle$ of H, the state $T|\psi\rangle$ is an eigenstate of H that is orthogonal to $|\psi\rangle$.

Solution: Let $|\psi\rangle$ be an eigenvector of H and assume H is invariant under time-reversal symmetry. First, we show that $T|\psi\rangle$ is an eigenstate of H with eigenvalue ε. The eigenstate satisfies

$$H|\psi\rangle = E|\psi\rangle \ .$$

Because, H is time-reversal symmetric, we know $TH = HT$ and therefore

$$HT|\psi\rangle = TH|\psi\rangle = \varepsilon T|\psi\rangle \ .$$

Next, we show $T|\psi\rangle$ is orthogonal to $|\psi\rangle$.

$$\begin{aligned} |\varphi\rangle &:= T|\psi\rangle \\ \langle\psi|\varphi\rangle &= (\langle T\psi|T\varphi\rangle)^* = \left(\langle T\Psi|T^2|\psi\rangle\right)^* \\ &= -(\langle\varphi|\psi\rangle)^* = -\langle\psi|\phi\rangle \quad \Rightarrow \quad \underline{\underline{\langle\psi|\varphi\rangle = 0}} \end{aligned}$$

End