

Topology in condensed matter physics

Exercise sheet 4

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4.1 Homotopies and contractible curves

Consider the continuous maps from the circle to the plane $f : \mathbb{S}_1 \rightarrow \mathbb{R}^2$.

1. (2 points) Show that all these maps f are point contractible. This means they are homotopic to the constant map $c : \mathbb{S}_1 \rightarrow \mathbb{R}^2$ with $x \mapsto p$ for an arbitrary, x -independent $p \in \mathbb{R}$.

Solution: Given a map $f : [0, 1] \rightarrow \mathbb{R}^2$, construct $H : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^2$ with

$$(s, t) \rightarrow f(s)|f(s)|(1-t).$$

We observe that this map is continuous because the absolute value function for positive, real numbers is continuous (even smooth) and $(1-t)$ is continuous (as well as the product of continuous functions). Furthermore, we have $H(s, 0) = f(s)$ and $H(s, 1) = (0, 0)$. By definition, H is a homotopy between f and the constant path $c : [0, 1] \rightarrow \mathbb{R}^2$ $s \rightarrow (0, 0)$. Hence, f is point contractible.

2. (1 point) Show that all these maps f are homotopic to each other. You have just shown that the fundamental group of the plane is trivial.

Solution: All maps are homotopic to $c : [0, 1] \rightarrow \mathbb{R}^2$, $s \rightarrow (0, 0)$ as shown above. The corresponding homotopy for a path f shall be called H_f . Then

$$H(s, t) = \begin{cases} H_f(s, 2t) & \text{if } t \leq \frac{1}{2} \\ H_g(s, 2-2t) & \text{if } t > \frac{1}{2} \end{cases}$$

is continuous because $H_f(s, \frac{1}{2}) = H_g(s, \frac{1}{2}) = (0, 0)$. Furthermore, $H(s, 0) = f(s)$, $H(s, 1) = g(s)$. Hence, g and f are homotopic.

4.2 Fundamental group

In this exercise, we complete the proof that the fundamental group is a group. In the following, we consider X to be a topological space and x_0 to be a fixed point in this space.

1. (1 point) Well-definedness. Show that the concatenation operation $[f] + [g] = [f + g]$ with $f, g : \mathbb{S}_1 \rightarrow X$ is “well-defined” for homotopy classes of loops. This is equivalent to saying that taking an $f_1 \in [f]$ and a $g_1 \in [g]$, it follows that $[f_1 + g_1] = [f + g]$.

Reminder: $f + g$ for $f, g : [0, 1] \rightarrow X$ with $f(0) = g(0) = f(1) = g(1) = x_0 \in X$ is defined as

$$(f + g)(s) = \begin{cases} f(2s) & \text{for } s \leq 1/2, \\ g(2s - 1) & \text{for } s > 1/2. \end{cases} \quad (1)$$

Solution: We construct the homotopy from $f + g$ to $f_1 + g_1$. First, we denote with H_{f,f_1} and $H_{g,g_1} : [0, 1] \times [0, 1] \rightarrow X$ the homotopies between f, f_1 and g, g_1 respectively. Then

$$H = \begin{cases} H_{f,f_1}(2s, t) & \text{if } s \leq \frac{1}{2} \\ H_{g,g_1}(2s - 1, t) & \text{if } s > \frac{1}{2} \end{cases}$$

is the homotopy from $f + g$ to $f_1 + g_1$. Hence, $[f_1 + g_1] = [f + g]$.

Checks:

$$H(s, 0) = \begin{cases} f_1(2s) & \text{if } s \leq \frac{1}{2} \\ g_1(2s - 1) & \text{if } s > \frac{1}{2} \end{cases} = f + g$$

$$H(s, 1) = \begin{cases} f_1(2s) & \text{if } s \leq \frac{1}{2} \\ g_1(2s - 1) & \text{if } s > \frac{1}{2} \end{cases} = f_1 + g_1$$

H is continuous:

$$\lim_{s \searrow \frac{1}{2}} H(s, t) = \lim_{s \searrow \frac{1}{2}} H_{g,g_1}(2s - 1, t) = x_0$$

$$\lim_{s \nearrow \frac{1}{2}} H(s, t) = H_{f,f_1}(1, t) = x_0$$

2. (1 point) parametrization unimportant. Show that loops of different parametrization that preserve the direction are homotopic. A path $f : I \rightarrow X$ is said to be of a different parametrization than $g : I = [0, 1] \rightarrow X$, if there is a homeomorphism $p : I \rightarrow I$ such that $f \circ p = g$. The orientation remains preserved if $p(0) = 0$ and $p(1) = 1$.

Solution: Consider $f : [0, 1] \rightarrow X$, $g : [0, 1] \rightarrow X$ with a suitable $p : [0, 1] \rightarrow [0, 1]$ (continuous bijection) such that $f = g \circ p$. Then $H : [0, 1] \times [0, 1] \rightarrow X$ with

$$(s, t) \rightarrow g(p(s)(1 - t) + ts)$$

is a homotopy for g and f .

Checks: H is continuous because only concatenations, sums, and products of cont. functions are involved.

$$H(s, 0) = g(p(s)) = f(s)$$

$$H(s, 1) = g(p(s) \cdot 0 + s) = g(s)$$

H is well defined because the appearing argument of g is a convex sum of elements of $[0, 1]$.

Check minimal values: $p(s) = s = 1 \Rightarrow p(s)(1 - t) + ts = 1$

Check maximal values: $p(s) = s = 0 \Rightarrow p(s)(1 - t) + ts = 0$

3. (1 point) neutral element. Show that every nonempty topological space has a point contractible loop.

Solution: The neutral element is $c : [0, 1] \rightarrow X$, $s \mapsto x_0$.

4. (1 point) associativity. Show formally that $[(f_1 + f_2) + f_3] = [f_1 + (f_2 + f_3)]$.

Solution:

$$\left((f_1 + f_2) + f_3 \right)(s) = \begin{cases} f_1(4s) & \text{if } s \leq \frac{1}{4} \\ f_2(2s - \frac{1}{2}) & \text{if } \frac{1}{4} < s \leq \frac{1}{2} \\ f_3(2s - 1) & \text{if } s > \frac{1}{2} \end{cases}$$

$$\left(f_1 + (f_2 + f_3) \right)(s) = \begin{cases} f_1(2s) & \text{if } s \leq \frac{1}{2} \\ f_2(4s - 2) & \text{if } \frac{1}{2} < s \leq \frac{3}{4} \\ f_3(4s - 3) & \text{if } \frac{3}{4} < s \end{cases}$$

This is just a reparametrization

$$p : [0, 1] \rightarrow [0, 1], \quad s \mapsto \begin{cases} 2s & \text{if } s \leq \frac{1}{4} \text{ double as fast} \\ s + \frac{1}{4} & \text{if } \frac{1}{4} < s \leq \frac{1}{2} \text{ same velocity} \\ \frac{1}{2}s & \text{if } s > \frac{1}{2} \text{ half as fast} \end{cases}$$

As we have shown before, reparametrizations of paths are homotopic.

4.3 Fermions and second quantization

Consider a system of spinful fermions (spin \uparrow and \downarrow) that can occupy a single orbital. This means the Hilbert space of this quantum system is spanned by the four states $|0\rangle$ (no fermion), $|\uparrow\rangle = c_{\uparrow}^{\dagger} |0\rangle$ (single fermion with spin up), $|\downarrow\rangle = c_{\downarrow}^{\dagger} |0\rangle$ (single fermion with spin down), $c_{\uparrow}^{\dagger} c_{\downarrow}^{\dagger} |\uparrow\downarrow\rangle$ (one fermion with spin up, one with spin down), where c_{σ}^{\dagger} with $\sigma \in \{\uparrow, \downarrow\}$ creates a fermion with spin σ . The fermionic anticommutation relations $\{c_{\sigma}^{\dagger}, c_{\sigma'}\} = c_{\sigma}^{\dagger} c_{\sigma'} + c_{\sigma'} c_{\sigma}^{\dagger} = \delta_{\sigma, \sigma'}$ and $\{c_{\sigma}, c_{\sigma'}\} = 0$ hold. The vacuum state $|0\rangle$ has the defining property that it is annihilated to zero by all annihilator operators $c_j |0\rangle = 0$. The Hamiltonian is

$$\mathcal{H} = \mu (n_{\uparrow} + n_{\downarrow}) + B (n_{\uparrow} - n_{\downarrow}), \quad (2)$$

where $n_{\sigma} = c_{\sigma}^{\dagger} c_{\sigma}$, μ is the on-site energy, and B corresponds to a magnetic field.

- (2 points) Show that $|0\rangle, |\uparrow\rangle, |\downarrow\rangle$, and $|\uparrow\downarrow\rangle$ are pairwise orthogonal. Also show that the states are normalized if the vacuum state is normalized.

Solution:

Case 1: $\langle 0|c_\sigma^\dagger|0\rangle = \underline{0}$, because we know $\langle 0|c_\sigma^\dagger = 0$ and $c_\sigma|0\rangle = 0$

Case 2: $\langle \sigma|\tilde{\sigma}\rangle = \langle 0|c_\sigma c_\sigma^\dagger|0\rangle$

$$\begin{aligned} & \left| \begin{aligned} c_\sigma c_\sigma^\dagger &= \delta_{\sigma\tilde{\sigma}} - c_\sigma^\dagger c_\sigma \end{aligned} \right. \\ &= \langle 0|\delta_{\sigma\tilde{\sigma}}|0\rangle - \langle 0|c_\sigma^\dagger c_\sigma|0\rangle \\ &= \underline{\underline{\delta_{\tilde{\sigma}\sigma}}} \end{aligned}$$

Case 3: $\langle \sigma_1|\sigma_2\sigma_3\rangle = \langle 0|c_1 c_2^\dagger c_3^\dagger|0\rangle$

$$\begin{aligned} &= \langle 0|(\delta_{12} - c_2^\dagger c_1) c_3^\dagger|0\rangle \\ &= \delta_{12} \underbrace{\langle 0|c_3^\dagger|0\rangle}_{=0} - \underbrace{\langle 0|c_2^\dagger c_1 c_3^\dagger|0\rangle}_{=0} \\ &= \underline{\underline{0}} \end{aligned}$$

Case 4: $\langle \sigma_1\sigma_2|\sigma_3\sigma_4\rangle = \langle 0|c_1 c_2 c_3^\dagger c_4^\dagger|0\rangle$

$$\begin{aligned} &= \langle 0|c_1(\delta_{23} - c_3^\dagger c_2) c_4^\dagger|0\rangle \\ &= \delta_{23} \langle 0|\delta_{14} - c_4^\dagger c_1|0\rangle - \langle 0|(\delta_{13} - c_3^\dagger c_1) c_2 c_4^\dagger|0\rangle \\ &= \delta_{23} \delta_{14} - \delta_{13} \langle 0|c_2 c_4^\dagger|0\rangle \\ &= \underline{\underline{\delta_{23} \delta_{14} - \delta_{13} \delta_{24}}} \end{aligned}$$

2. (2 points) Express the Hamiltonian in matrix form

$$\mathcal{H} = \sum_{j,k} c_j^\dagger h_{j,k} c_k, \quad (3)$$

i.e., find out the coefficients $h_{j,k}$.

Solution:

$$\mathcal{H} = \begin{pmatrix} c_\uparrow^\dagger & c_\downarrow^\dagger \end{pmatrix} \begin{pmatrix} \mu + B & 0 \\ 0 & \mu - B \end{pmatrix} \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix}$$

3. (2 points) Find out the spectrum (the eigenvalues) of the matrix h and compare them to the spectrum of \mathcal{H} . Explain how to construct the spectrum of \mathcal{H} from the spectrum of h .

Solution:

$$\varepsilon_1 = \mu + B, \quad \varepsilon_2 = \mu - B$$

$$\mathcal{H}|0\rangle = 0$$

$$\mathcal{H}|\uparrow\rangle = (\mu + B)|\uparrow\rangle$$

$$\mathcal{H}|\downarrow\rangle = (\mu - B)|\downarrow\rangle$$

$$\mathcal{H}|\uparrow\downarrow\rangle = (2\mu)|\uparrow\downarrow\rangle$$

$$\Rightarrow E_1 = 0, \quad E_2 = \mu + B, \quad E_3 = \mu - B, \quad E_4 = 2\mu$$

The Hamiltonian in the Fockspace is a 4×4 diagonal matrix. The spectrum is then given by the eigenvalues.

4. (2 points) Discuss if there are off-diagonal terms of the Hamiltonian \mathcal{H} that physically make sense.

Solution: If we would consider off-diagonal terms, we have the following two cases.

For the **one-particle basis**, off-diagonal terms describe the coupling of states as a result of for instance an external magnetic field in the $x - y$ plane.

For the **Fock space basis** (second quantization), they would describe processes, that would not conserve the particle numbers. This is only possible for instance in the BCS theory (cooper-pairs) and specific transformations in interacting systems as in bosonization processes (transforming a system of interacting fermions into a massless system of not-interacting bosons).

End