Topology in condensed matter physics
Exercise sheet 3

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3.1 Separation axioms

Consider the following so-called separation axioms for topological spaces, which are properties that a topological space may or may not have.

T0 (Kolmogorov) For each pair of distinct points there is an open set that contains one of the points but not the other one (the points are topologically distinguishable).

T2 (Hausdorff) For each pair of distinct points there are two disjoint open sets, each containing exactly one of the points.

T4 (normal Hausdorff) For each pair of disjoint, closed sets there are two disjoint open neighborhoods of these sets, each containing exactly one of the closed sets.

For further discussion: Separation axioms

(T0) *Kolmogrov space*: For every pair of distinct points $x, y \in \mathcal{X}$, with $x \neq y$, exists at least one neighbourhood U, which contains one of the points $x \in U$ but not the other one $y \notin U$.
(T1) *Accessible/ Frechet* For every two points $x, y \in X$, with $x \neq y$, exists a neighbourhood $x \in U$ and $y \in V$ for each point, which does not contain the other point $x \neq U, y \neq V$. The neighbourhoods U and V do not need to be disjoint. (T2) *Hausdorff:* For every two points $x, y \in X$, with $x \neq y$, exists a neighbourhood $x \in U$ and $y \in V$ for each point, which does not contain the other point $x \neq U, y \neq V$. The neighbourhoods are disjoint $U \cap V = \emptyset$.

(T3) *Regular Hausdorff:* For every point $x \in \mathcal{X}$ and every closed set $A \in X$, with $x \notin A$, exists a disjoint neighbourhood $x \in U$ and $A \subset V$. The neighbourhoods are disjoint $U \cap V = \emptyset$, with $U, V \in X$.

(T4) *Normal Hausdorff:* For every two disjoint closed subsets $A \cap B = \emptyset, \quad A, B \subset X$, exist neighbourhoods $A \subset U$ and $B \subset V$, which are disjoint $U \cap V = \emptyset, \; U, V \in X$.

Furthermore, we can conclude:

- Every Hausdorff space also satisfies (T1)

$$(T_2) \;\Rightarrow\; (T_1)$$

- We can not directly get T_2 or T_3 from T_4

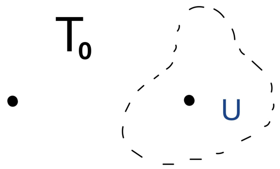
$$(T_4) \not\Rightarrow (T_3) \not\Rightarrow (T_2) \ .$$

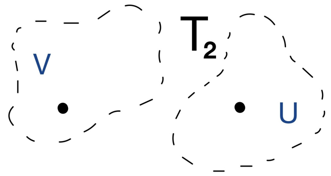
- A Hausdorff space that satisfies (T_4) also satisfies (T_3)

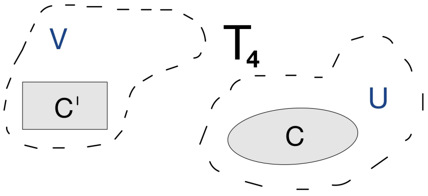
$$(T_4) \wedge (T_2) \;\Rightarrow\; (T_3) \ .$$

1. Draw a schematics for the separation axioms $T0$, $T2$, and $T4$.

Solution:





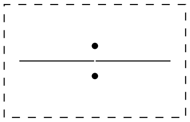


2. Which separation axioms are valid for the following topological spaces?
   (a) $\mathbb{R}^2$

Solution:

Valid: $T_0,\ T_2,\ T_3,\ T_4$ see intuitive proofs by examples above

   (b) $X/\sim$, where $X = \mathbb{R}\times a \cup \mathbb{R}\times b$ and $\sim$: $(x,a)\sim(x,b)$, if $x \neq 0$. *schematics*:



Solution:

Note: Everywhere else than $(0,a)$ & $(0,b)$, the space is locally homeomorphic to $\mathbb{R}$. Hence, only the points $(0,a)\&(0,b)$ have to be regarded here.
T_0 : Points $(0,a),\ (0,b)$ have a separating neighbourhood, e.g., $U = \{[x,a]:\ x \in (-1,1)\}$.
T_2 : Every neighbourhood of (a,b) intersects with every neighbourhood of $(0,b)$. Hence, T_2 is not valid.
T_3/T_4 : $\{(0,a)\}$and$\{(0,b)\}$ are closed sets. Hence, T_3 is valid but not T_4.

   (c) The graph of $f:\mathbb{R}\to\mathbb{R}$, $x\mapsto \sin(1/x)$ as a subspace of $\mathbb{R}^2$.

Solution:

The graph of $f(x) = \sin(\frac{1}{x})$ is a subspace of $\mathbb{R}^2$ and is not defined for $x = 0$.
$T_0/T_2/T_4$: Two distinct points have different x coordinates. We can always find disjoint open sets and we can always separate disjoint closed sets with disjoint open sets. The graph, therefore, satisfies T_4, T_2 and T_0.

(d) $(X, \mathcal{T})$ with $X = \{a, b, c\}$, $\mathcal{T} = \{\{a, b, c\}, \emptyset, \{a\}, \{a, b\}\}$.

Solution:

T_0 : Consider each pair of points and find a separating neighbourhood:

$$(a, b) \Rightarrow U = \{a\} \qquad (c, a) \Rightarrow U = \{a\}$$
$$(b, c) \Rightarrow U = \{a, b\} \qquad \Rightarrow T_0 \text{ is fulfilled}$$

T_2 : Consider (b, c) . The only neighbourhood of c is $\{a, b, c\}$. Hence, T_2 is not fulfilled.
T_3 : $\{c\}$ is a closed set. The only neighbourhood is $\{a, b, c\}$. Hence, T_3 is not fulfilled.
T_4 : $\{c\}$ and $\{c, b\}$ are closed. There is no disjoint neighbourhood of these sets.

(e) $(X, \mathcal{T})$ with $X = \{a, b, c\}$, $\mathcal{T} = \{\{a, b, c\}, \emptyset\}$.

Solution:

The topology of (e) is coarser than the one of (d). Therefore every separation axiom that is not fulfilled in (d) is also not fulfilled in (e). The remaining one is
T_0 : The only nonempty open set is $\{a, b, c\}$. Hence, no point has a neighbourhood in which another point is not included, which shows T_0 is not fulfilled.

Definition of a manifold

For your information: a Hausdorff space (i.e., a topological space fulfilling T_2), which is locally homeomorphic to $\mathbb{R}^n$ for some positive integer n and has a countable base for its topology is called **manifold**. Manifolds are of central importance to the mathematical description of physics. Almost every space you have encountered during your studies can be seen as a manifold, including finite dimensional vector and Hilbert spaces, the phase space, and, of course, spacetime.

3.2 Representations of topological spaces

Draw (embeddings of) the following topological spaces. Every time there is no topology specified, assume the subspace topology of $\mathbb{R}^n$.

Homeomorphism

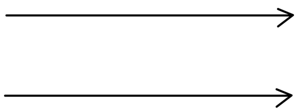
A *homeomorphism* is a bijective map $f : (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ between topological spaces, where the map f and the inverse map f^{-1} is continuous.
If there is a homeomorphism, the topological spaces are called *homeomorph* to each other.

In the following exercise, we depict spaces that are homeomorphic to each other.

1. (1 point) $\mathbb{R} \setminus \{0\}$;

Solution:

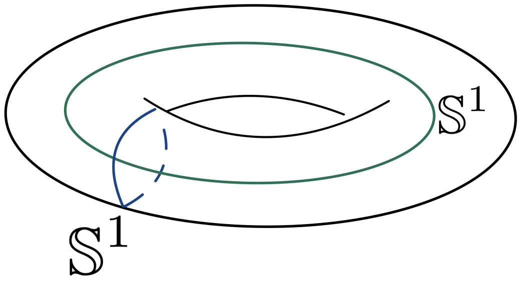
The point 0 is not included.



2. (1 point) $\mathbb{S}^1 \times \mathbb{S}^1$, where $\mathbb{S}^1$ is the circle;

Solution:

The product of the topological spaces $\mathbb{S}^1 \times \mathbb{S}^1$ gives the $\mathbb{T}^2$ Torus.



Torus (pl.: Tori)

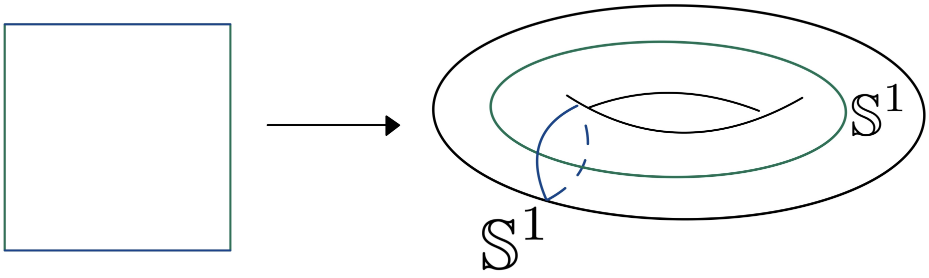
In general, the torus is defined as

$$\mathbb{T}^n = \overbrace{\mathbb{S}^1 \times ... \times \mathbb{S}^1}^{n} .$$

3. (1 point) $X/\sim$, where $X = [0,1] \times [0,1]$ and $x \sim y$ if (and only if) 1) $x = y$, 2) $x = (x_1, 1)$ and $y = (x_1, 0)$ for all $x_1 \in [0,1]$ or 3) $x = (1, y_1)$ and $y = (0, y_1)$ for some $y_1 \in [0,1]$;

Solution:

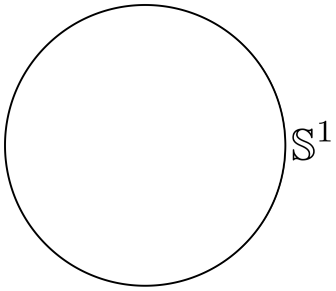
The product of $[0,1] \times [0,1]$ is a square. Identifying the upper line and the lower line as the same, with the same orientation, results in a cylinder. Doing the same for the other two lines leads again to the T^2 Torus.



4. (2 points) $X/\sim$, where $X = \mathbb{R}^2 \setminus \{(0,0)\}$ and $x \sim y$ if (and only if) $x = y$ or if the line joining x and y would go through the origin $\{(0,0)\}$. This space is an example of a projective space.

Solution:

We can start with $\mathbb{R}$, without the origin. The space includes the set of lines that go through the origin. We first project all points on $\mathbb{S}^1$. Gluing then points to the opposite sites on the same line leads to a figure eight. Doing the same for the rest of the point leads to $\mathbb{S}^1$.



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