Topology in condensed matter systems
Exercise sheet 2

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2.1 Metric spaces

Let (X, d) be a metric space.

1. (1 point) Show that the positivity $d(x, y) \geq 0$ of a metric follows from the remaining three axioms, which are symmetry, the identity of indiscernibles, and the triangle inequality.

Solution:

A *metric space* (X, d) is a set X with a function

$$d: \; X \times \; X \to \mathbb{R}$$

with the properties (M1) - (M3) and $x, y, z \in X$.

(M1) Positive definite:

$$\forall x, y \Big(\overbrace{d(x,y) \geq 0}^{\text{M1.A: Positivity}} \quad \wedge \quad \overbrace{d(x,y) = 0 \Leftrightarrow x = y}^{\text{M1.B: Identity of indiscernibles}} \Big)$$

(M2) Symmetry:

$$\forall x, y \Big(\; d(x,y) = d(y,x) \; \Big)$$

(M3) Triangular inequality:

$$\forall x, y, z \Big(\; d(x,y) \leq d(x,z) + d(z,y) \; \Big)$$

Claim: The positivity axiom for a metric M1.A follows from the identity of indiscernibles M1.B, the symmetry axiom M2 and the triangular inequality axiom M3 with the addition of knowing that $d(x, x) = 0$.

Proof: Let d be a metric on the set X, such that $d : X \times X \to \mathbb{R}$, and $x, y, z \in X$.
Let w.l.o.g, x be equal to y and with $d(x, x) = 0$ it follows that

$$0 \leq d(y,z) + d(z,y) \; .$$

Using the symmetry axiom M2 results in

$$0 \leq 2 \cdot d(z,y) \; .$$

This shows $0 \leq d(z, y)$ and therefore also $0 \leq d(x, y)$. □

2. (2 points) Show that the identity of indiscernible follows for a metric space follows from the axioms $d(x,y) > 0$ for $x \neq y$, $d(x,x) = 0$, and the triangle inequality. Identity ofo indiscernibles:

$$d(x,y) = 0 \Leftrightarrow x = y \ .$$

3. (2 points) An open set $U \subset X$ is a set with all elements $x \in U$ having an open ball that contains x inside U. Show that an open set in a metric space is the union of open balls.

Solution:

Claim: Let U be an open set $U \subset X$ and (X,d) a metric space, then we can write

$$U = \cup_{x \in U} B_{r_x}(x) \ .$$

Proof: By construction, we know that $\cup_{x \in U} B_{r_x}(x) \subset U$. By the definition of openness, there exists $r_x > 0$ with $B_{r_x}(x) \in U$. This shows $\forall x \in U$ that

$$x \in B_{r_x}(x) \quad \Rightarrow \quad x \in \cup_{x \in U} B_{r_x}(x)$$

and therefore, $U \subset \cup_{x \in U} B_{r_x}(x)$. This shows $U = \cup_{x \in U} B_{r_x}(x)$. $\square$

4. (2 points) Show that the "induced topology" of a metric space (as defined in the lecture) is a topology. *Reminder:* The natural topology of a metric space includes all sets that are unions of open balls and the empty set (or balls of radius $r = 0$ allowed).

Solution: To show that the natural topology of a metric space (X, d) is a topology, we need to show that it

1. includes the empty set: Yes, because $\emptyset = B_0(x)\ \forall x \in X$
2. includes the full set: Yes, because $X = \cup_{r\in\mathbb{R}} B_r(x) \forall x \in X$
3. includes all unions of its elements: Yes, by definition.
4. includes finite sections: Finite sections are reducible to sections of pairs. Consider two sets that are unions of (open) balls $U_1 = \cup_{i\in I} B_i$ and $U_2 = \cup_{j\in J} B'_j$ with some index sets I & J. Then $U_1 \cap U_2 = \cup_{i\in I, j\in I} \left(B_i \cup B'_j \right)$. Hence, if the intersection of two open balls is representable as a union of open balls, so is every finite intersection of sets in the natural topology. We hence regard $B_{r_y}(y)$ & $B_{r_x}(y)$ with $r_x, r_y \in \mathbb{R} \geq 0$ & $x, y \in X$ and realize that $B(z)_{\min([r_z - d(z,y)],[r_x - d(z,x)])}$ is a ball, which is inside $B_{r_y}(y) \cap B_{r_x}(x)$ and not empty if $z \in B_{r_y}(x) \cap B_{r_x}(x)$. Hence, $\cup_{z\in B_{r_y}(y)\cap B_{r_x}(x)} B(z) = B_{r_y}(y) \cap B_{r_x}(x)$. It follows from what was mentioned before that finite intersections of elements of the natural topology are contained in the natural topology and hence, the natural topology is a topology. □

5. (2 points) Construct a set that is not an open set but can be represented as the intersection of open sets. *Hint: Think, e.g., about infinite sections of open balls in a metric space.*

Solution:

Claim: The set $A = \cap_{n=1}^{\infty} \mathcal{O}_n$, with $\mathcal{O}_n = (\frac{-1}{n}, \frac{1}{n})$, is not an open set but is an infinite intersection of open sets.

Proof: The set $\mathcal{O}_n = (\frac{-1}{n}, \frac{1}{n})$, $n \in \mathbb{N}$ is open in $\mathbb{R}$. The limits are

$$\lim_{n\to\infty} \frac{-1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

and for all elements of the interval with $\frac{-1}{n} < x < \frac{1}{n}$ it follows with

$$\left(\lim_{n\to\infty} \frac{-1}{n} = 0 \right) < \lim_{n\to\infty} x < \left(\lim_{n\to\infty} = 0 \right) ,$$

the limit $\lim_{n\to\infty} x = 0$, and therefore

$$A = \cap_{n=1}^{\infty} \mathcal{O}_n = \{0\} .$$

The complement

$$\mathbb{R} \backslash \{0\} = (-\infty, 0) \cup (0, \infty)$$

is open. This shows the set A is closed. □

2.2 Connection between equivalence relation and partition

An equivalence relation $\sim$ on a set X can be defined as a map $\sim: X^2 \to \{\text{true}, \text{false}\}$ with the following properties[1]. Let $x, y, z \in X$, then:

reflexivity $x \sim x$.

symmetry If $x \sim y$, then $y \sim x$.

transitivity If $x \sim y$ and $y \sim z$, then $x \sim z$.

An equivalence class $[x]$ for $x \in X$ is the subset of all elements of X, that are equivalent to x.

1. (2 points) Consider a partition $\mathcal{P}$ of X. That is a family of disjoint subsets of X, the union of which is X. A partition induces a relation between two elements of X by $\sim: X^2 \to \{\text{true}, \text{false}\}$ by $x \sim y$ exactly then, if $\mathcal{P}$ contains a set, which possesses x as well as y. Show that this relation is an equivalence relation

Solution:

Claim: A partition $\mathcal{P}$ induces an equivalence relation by $x \sim y$ iff there is a set in $\mathcal{P}$ that contains both, x & y .
Proof:

1. Reflexivity: $x \sim x$ because there is a set $A \in \mathcal{P}$ with $x \in A$ because, $\cup\mathcal{P} = X$. $\left(\cup_{A \in \mathcal{P}} A = X\right)$
2. Symmetry: If $x \sim y$ then there is a set $A \in \mathcal{P}$ which contains x & y . Hence, also $y \sim x$.
3. Transitivity: If $x \sim y$ & $y \sim z$, then there is a set $A \in \mathcal{P}$ with $x, y, z \in A$. Hence, $x \sim z$ $\square$

2. (1 point) Show that the family of equivalence classes of an equivalence relation constitutes a partition.

Solution:

Claim: The family of equivalence classes of an equivalence relation constitutes a partition.

Proof: Let $\sim$ be our equivalence relation on X and $X \neq \emptyset$. We can use the reflexivity. Consequently for $x \sim x$ it follows that $x \in [x]$ and therefore $[x] \neq \emptyset$.
Now, we show, that every element of X belongs to exactly one equivalence class. From the reflexivity, we know that $x \in X$ belongs to at least one class. Let $x \in [y]$ and $x \in [z]$, then $x \sim y$ and $x \sim z$ and with the symmetry and transitivity, it follows that $y \sim z$. This shows $[y] = [z]$. Therefore, x belongs to exactly one equivalence class. $\square$

You have just shown that an equivalence relation can be interpreted as a partition and vice versa.

End

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[1] Usually one writes $x \sim y$ *(read "x is equivalent to y")* instead of "$\sim (x, y)$ *is true* ".