Topology in condensed matter physics
Exercise sheet 12

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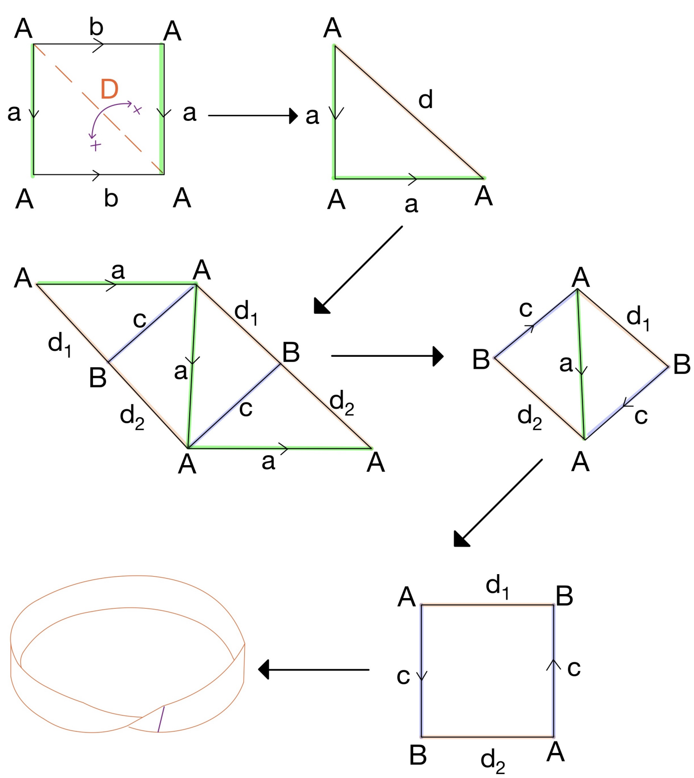
12.1 Two indiscernible particles on a ring

Derive the configuration space for two indiscernible particles on a ring $\mathcal{C}_2^{\mathbb{S}_1} = ((\mathbb{S}_1 \times \mathbb{S}_1 \setminus D)/S_2$ by, for instance, repeating the steps from the lecture that led to the configuration space of two particles in $1D$, $2D$, and $3D$. As a reminder: here $D = \{(a,a)|a \in \mathbb{S}_1\}$ and S_2 is the symmetric group with two elements (i.e., permutations of two elements).

Solution:

First draw $\mathbb{S}^1 \times \mathbb{S}^1$. This is a Torus which we can represent in a gluing diagram. Further, we exclude the diagonal D and identify same states. This leaves a triangle.
While this description already is sufficient, we can transform it into a somewhat better known one.

With drawing two copies of $\mathcal{C}_2^{\mathbb{S}^1}$ with the additional point B and the segments d_1, d_2, c we find that $\mathcal{C}_2^{\mathbb{S}^1}$ is homeomorphic to the Möbius strip.



12.2 Linear representations up to a phase

When developing the quantum theory for non-simply connected spaces, a one-dimensional, complex "representation up to a phase" of the fundamental group of the space in question is needed. Consider X to be a path-connected topological space. Such a representation $\gamma : \pi_1(X) \to \mathbb{C}$ needs to fulfill

$$\gamma(f+g) = \gamma(f)\gamma(g)e^{\mathrm{i}\phi}, \tag{1}$$
$$|\gamma(g)| = 1. \tag{2}$$

with some constants $\phi \in [0, 2\pi)$ for all $f, g \in \pi_1(X)$.

a. Show that $\gamma : \mathbb{Z} \to \mathbb{C}$ with $\gamma(z) = e^{\mathrm{i}(\varphi z - \phi)}$ fulfills the above requirements. These are possible representations used in the quantum mechanical description of 2 anyons in 2D or for one particle on a ring. The parameter $\varphi \in [0, 2\pi)$ is called the statistical angle and describes the phase a particle picks up when traveling a path corresponding to the group element 1.

Solution:

$$\gamma(f+g) = e^{i(\varphi(f+g)-\phi)} = e^{i(\varphi f - \phi)} e^{i(\varphi g - \phi)} e^{i\phi} \underline{\underline{= \gamma(f)\gamma(g)e^{i\phi}}}$$
$$|\gamma(g)| = |e^{i(\phi g + \phi)}| = \sqrt{\cos^2(\phi g + \phi) + \sin^2(\phi g + \phi)} = \underline{\underline{1}}$$

b. Show that for $\pi_1(X) = \mathbb{Z}_2$, i.e., the group with exactly two elements, there are no other such representations than $\gamma_1, \gamma_2 : \mathbb{Z}_2 \to \mathbb{C}$ with $\gamma_1(z) = 1$ and $\gamma_2(z) = (-1)^z$ (up to an additionally multiplied phase factor). This is to say: in three dimensions, there are only bosons and fermions, but no anyons.

Solution: A representation $\gamma : \left[\mathbb{Z}_2 = (a,b)\right] \to \mathbb{C}$ is of the form $\gamma(a) = e^{i\varphi_a},\ \gamma(b) = e^{i\varphi_b}$ because of the $|\gamma(g)| = 1$. If we now apply the first identity to all possibilities, we obtain

$$\gamma(a+a) = \begin{cases} = \gamma(a) = e^{i\varphi a} \\ \gamma(a)\gamma(a)e^{i\phi} = e^{i2\varphi_a}e^{i\phi} \ \Rightarrow\ e^{i\varphi_a + i\phi} = a \end{cases} \quad \text{(a)}$$
$$\gamma(b+b) = \Big\{ \gamma(a) = e^{i\varphi_a} = \gamma(b)\gamma(b)e^{i\phi} = e^{i2\varphi_b}e^{i\phi} \ \Rightarrow\ e^{i\varphi_a - 2i\varphi_b - i\phi} = 1 \quad \text{(b)}$$
$$\gamma(a+b) = \gamma(b+a) = \begin{cases} = \gamma(b) = e^{i\varphi_b} \\ = \gamma(b+a) = \gamma(a)\gamma(b)e^{i\phi} = e^{i\varphi_b}e^{\varphi_a}e^{i\phi} \ \Rightarrow\ e^{i\varphi_a + i\phi} \text{ (c)=(a)} \end{cases}$$

By (a)/(b), we have $e^{2i(\varphi_b+\phi)} = 1$. This results in $\varphi_b + \phi \in \{0, \pi\}$ and $\varphi_a + \phi \in \{0\}$ without loss of generality because φ_a & φ_b are defined modulo 2π only.
Hence there are only two possibilities
$\gamma_1 : \gamma_1(a) = e^{i\phi},\ \gamma_1(b) = e^{i\phi}$ and
$\gamma_2 :\ \gamma_2(a) = e^{i\phi},\ \gamma_2(b) = -e^{i\phi},$ which corresponds exactly to the representations given on the exercise sheet.

End