Topology in condensed matter physics
Exercise sheet 11

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11.1 Quantum physics of indiscernible particles

a. (1 point) Find at least one mistake in the following textbook like the derivation of the existence of only bosons (symmetric wave function) and fermions (anti-symmetric wave function), which is still frequently printed in textbooks and articles, [see, e.g., Nature **637**, 314 (2025) Eqs. (1) and (2)]:
Consider the wave function of an N particle system of indiscernible particles (for simplicity without spins and additional degrees of freedom) in spatial representation $\Psi : \mathbb{R}^{N\times d} \to \mathbb{C}$, where d is the dimension of space. A permutation of two particles, x_i and x_j $(i \neq j)$, is represented by the exchange of the respective coordinates in the wave function. Then, in general,

$$\Psi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = c\Psi(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n). \tag{1}$$

We now exchange x_i and x_j on the right-hand side again and obtain

$$\Psi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = c^2\Psi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n). \tag{2}$$

We conclude that $c = 1$ and $c = -1$ are the only possibilities for c.

Solution:

The exercise sheet shows a “proof” that only antisymmetric wave functions or symmetric wave functions exist. There is a variety of mistakes as

- c does not depend on the coordinates a priory
- no physical assumption is needed
- it is assumed that the quantum theory of discernible particles can be used to describe the quantum theory of indiscernible ones
- what does “permutation” mean?
- why is “permutation” defined as the exchange of variables and why does this matter?
- ambiguity of labels

1. Claim: A permutation of particles is represented by exchanging the respective coordinates.
Proof: The particles x_i and x_j are indistinguishable. They have the same intrinsic properties. Let x_i and x_j be fermions and because they are indistinguishable, we can just relable them, what leads to

$$\Psi(x_1, x_2) = \Psi(x_2, x_1)\ .$$

However, we know fermions have an antisymmetric wave function. $\lightning$

2. Claim:

$$\Psi(x_1, x_2) \stackrel{?}{=} c\psi(x_2, x_1)$$

Proof: Let A be an unitary operator, U an antiunitary operator and K the complex conjugation operator. By applying A, it follows that

$$A\Psi(x_1, x_2) = KU\Psi(x_1, x_2) = c^*\Psi(x_2, x_1)$$

antiunitary symmetries were not considered. $\lightning$

3. Claim: $c = 1$ and $c = -1$ are the only possibilities (for $c^2 = 1$)

Proof:

In principle c could be a complex number. Consider picking up c for the permutation and c^* for permuting back.

$$\begin{aligned}\Psi(x_1, x_2) &= e^{i\phi}\Psi(x_2, x_1)\\ &= \begin{cases} \Psi(x_2, x_1) & \text{for } \phi = 2\pi n\\ -\Psi(x_2, x_1) & \text{for } \phi = (2n+1)\pi\\ i\Psi(x_2, x_1) & \text{for } \phi = \frac{\pi}{2} + 2\pi n\\ -i\Psi(x_2, x_1) & \text{for } \phi = \frac{3\pi}{2} + 2\pi n\\ \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\Psi(x_2, x_1) & \text{for } \psi = \frac{\pi}{4} + 2\pi n\\ \ldots & \\ \Big(\cos(\phi) + i\sin(\phi)\Big)\Psi(x_2, x_1) & \text{for } \phi \in \mathbb{R} \end{cases}\end{aligned}$$

Therefore, $c = \pm 1$ are in general not the only possibilities.

4. Claim: The domain of the wave function is $\mathbb{R}^{N\cdot d}$.

Proof: The domain $\mathbb{R}^{N\cdot d}$ suggests that, for example, for $N = 2$ and $d = 1$, we consider the whole space.
However, x_1 and x_2 are indistinguishable. This means the states (x_1, x_2) and (x_2, x_1) are the same. With $\mathbb{R}^{N\cdot d}$, we would therefore generate too many states. $\lightning$

Instead, use

$$C_N^d = (\mathbb{R}^{N\cdot d} - D)/\sim \quad .$$

11.2 Fun with braids

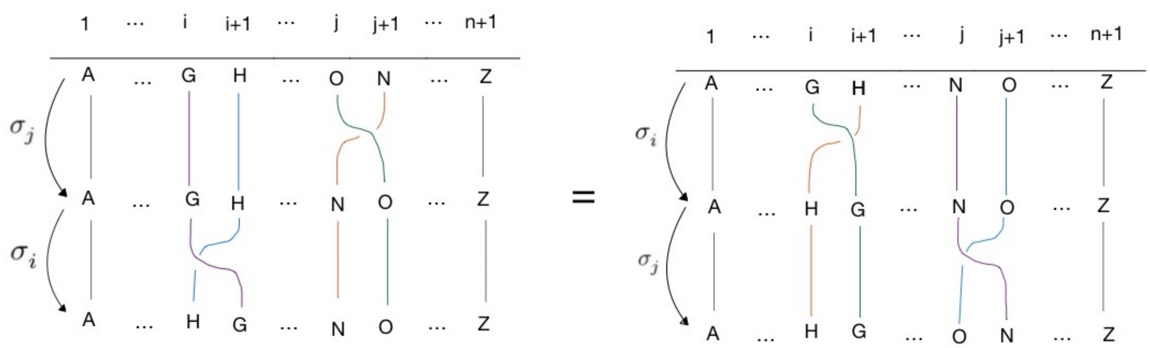
The braid group of order $n+1$ is generated by σ_i, where $i \in \{1 \ldots n\}$, which obey the following identities

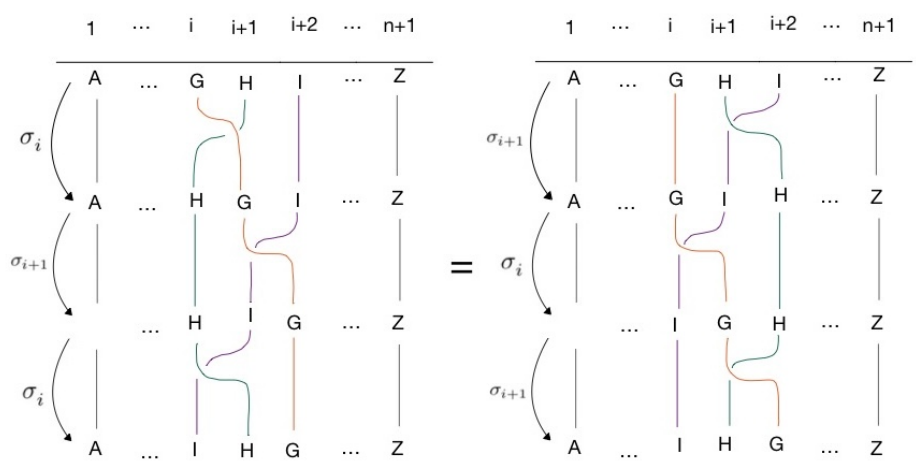
$$\left(\sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1}\right) = 1 \quad \text{if } |i-j| > 1, \tag{3}$$

$$\left(\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1}\right) = 1. \tag{4}$$

a. (2 points) Draw these relations with a “braid diagram”.

Solution:





b. (2 points) Set $\Delta = \sigma_1 \sigma_2 \sigma_1$. Show that

$$\sigma_1 \Delta = \Delta \sigma_2, \tag{5}$$

$$\sigma_2 \Delta = \Delta \sigma_1, \tag{6}$$

and hence that Δ^2 commutes with both σ_1 and σ_2.

Solution:

$$\sigma_1\Delta = \sigma_1\sigma_1\sigma_2\sigma_1$$

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$

$$= \sigma_1\sigma_2\sigma_1\sigma_2$$

$$\underline{\underline{= \Delta\sigma_2}}$$

$$\sigma_2\Delta \;=\; \sigma_2\sigma_1\sigma_2\sigma_1 \;=\; \sigma_1\sigma_2\sigma_1\sigma_1 \;\underline{\underline{= \Delta\sigma_1}}$$

$$\sigma_1\Delta^2 \;=\; \Delta\sigma_2\Delta \;\underline{\underline{= \;\Delta^2\sigma_1}}$$

$$\sigma_2\Delta^2 \;=\; \Delta\sigma_1\Delta \;\underline{\underline{= \;\Delta^2\sigma_2}}$$

11.3 Configuration space — alternative view

The configuration space of indiscernible particles has been defined in the lecture by taking the set of tuples in $\mathbb{R}^d$, where d is the spatial dimension, excluding tuples that include equal points and identifying tuples that result as permutations from each other. Here, we want to show that a different philosophy leads to the same result.

If n discernible particles are described, they can be labeled and hence their positions be collected into an n-tuple. If indiscernible particles shall be described, the labels are obsolete but still the n positions can be collected into a set. Sets naturally forget about the ordering of their elements, and naturally treat equal elements as one. Hence, consider the family $\mathcal{F}$ of sets with n elements, each element belonging to $\mathbb{R}^d$, where $d \in \{1, 2, 3\}$ is the dimension of space. For notational purposes, we also introduce the tuples with n pairwisely different elements $\mathcal{G} = \{x \in \left(\mathbb{R}^d\right)^n \mid \text{ with } x_j \neq x_i \text{ if } j \neq i\}$.

Now regard the map $q : \mathcal{G} \to \mathcal{F}$, which forgets about the order, i.e., $q\,(x_1, x_2, \ldots, x_n) = \{x_1, x_2, \ldots, x_n\}$. Take $\delta_x = \min_{i,j\in\{1,\ldots,n\}, i\neq j} \frac{1}{\sqrt{2}}\,(|x_i - x_j|)$ to be the minimal distance between two positions.

a. (1* points) Show that $q^{-1}(q(B_\delta(x)))$, where $x = (x_1, \ldots, x_n)$ is a tuple with pairwise different elements, is a direct sum of homeomorphic sets.

Solution:

The inverse map $q^{-1}\Big(q(B_\delta(x))\Big)$ decomposes into sets with all possible permutations of the n-tupel

$$q^{-1}\Big(\{x_1, ..., x_n\}\Big) = \; \Big\{x_{\sigma(1)}, ..., x_{\sigma(n)} : \sigma \in S_n\Big\} \;,$$

and therefore

$$\Rightarrow \; q^{-1}\Big(q(B_\delta(x))\Big) = \bigcup_{\sigma\in S_n} B_{\delta,\sigma}(x) \;.$$

Each set has per construction a unique permutation, so we can write for two permutations $\sigma \neq \sigma'$

$$B_{\delta\sigma} \cap B_{\delta,\sigma'} = \emptyset \;\Rightarrow\; q^{-1}\Big(q(B_\delta(x))\Big) = B_{\delta_{\sigma_1}}(x) \sqcup ... \sqcup B_{\delta_{\sigma_n}}(x) \;.$$

(To-Do: show that the distance is larger than 2δ)

To show, that the sets are homeomorphic, we can construct

$$f\Big(x_{\sigma(1)}, ..., x_{\sigma(n)}\Big) = \Big(x_{\sigma'(1)}, ..., x_{\sigma'(n)}\Big) \ ,$$

where f is a relabeling of the elements. The cardinality stays the same, what shows f is bijective and because we just relabel, f is also continuous and therefore a homeomorphism. This show the sets are homeomorph to each other.

We furthermore take $\mathcal{U} = \{q(B_{\gamma_x}(x)) \mid x \in \mathcal{G}, \gamma_x \in \mathbb{R} < \delta_x\}$ as a base to define the topology of $\mathcal{F}$. I.e., the topology $\mathcal{T}$ of $\mathcal{F}$ is constructed by all possible unions of elements of $\mathcal{U}$.

b. (2* points) Show that the so defined topology of $\mathcal{F}$ is the finest topology such that q is continuous. Subsequently show that therefore there is an isomorphism between the notion of a configuration space as defined in the lecture and as defined here.

Solution: The map q is the quotient map

End