Topology in condensed matter physics
Exercise sheet 10

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10.1 Adiabatic time evolution

Adiabatic time evolution is described by a linear, first-order differential equation for operators on a Hilbert space $\mathcal{H}$, which reads

$$K'(s) = [P'(s), P(s)]\, K(s), \tag{1}$$

where $P(s)$ is a smoothly varying projector onto a subspace of $\mathcal{H}$. Usually, this subspace is thought of as the instantaneous ground space of some Hamiltonian H. The solution Eq. (1) is chosen to fulfill the initial condition $K(0) = P(0)$.

a. Show that $K(s) = P(s)\Omega(s)$, where $\Omega(s) = \mathcal{T}\left\{e^{\int_0^s ds' A(s')}\right\}$, is a solution of Eq. (1). From the lecture, we know that $K(s) = \Omega(s)P(0)$ is also a solution of Eq. (1). In fact, we have $P(s)\Omega(s) = \Omega(s)P(0)$, which results in $P(s)K(s) = K(s) = K(s)P(0)$ and the time evolution of the projection operator $P(s) = \Omega(s)P(0)\Omega^{-1}(s)$.

Solution:

$$k' = \frac{d}{ds}P(s)\Omega(s)$$

$$\Bigg| \quad P(s)K(s) = K(s)$$

$$= \frac{d}{ds}P(s)P(s)\Omega(s)$$

$$= P'K(s) + P(s)P'(s)\Omega(s) + P(s)P(s)\Omega'(s)$$

$$\Bigg| \quad \Omega'(s) = [P'(s)P(s) - P(s)P'(s)]\Omega(s), \;\; P(s)P(s) = P(s)$$

$$= P'(s)P(s)K(s) \cancel{+ P(s)P'(s)\Omega} + P(s)P'(s)K(s) \cancel{- P(s)P'(s)\Omega(s)}$$

$$\Bigg| \quad P(s)P'(s)P(s) = -P(s)P'(s)P(s) = 0$$

$$\underline{\underline{= A(s)K(s)}}$$

10.2 Topological quantum computing with Majoranas

In this exercise, you show how the degenerate ground state in systems with Majorana modes can be adiabatically manipulated. Consider four Majorana operators γ_j with $j \in \{1 \ldots 4\}$ in a system with tunable on-site potentials

$$\mathcal{H}(t) = \mathrm{i}\left(\mu_b(t)\gamma_3\gamma_4 + \mu_r(t)\gamma_2\gamma_3 + \mu_l(t)\gamma_1\gamma_3\right). \tag{2}$$

a. (2 points) The adiabatic evolution of the Hamiltonian starts at $t = 0$ with $\mu_b(0) = \mu > 0$ and $\mu_l(0) = \mu_r(0) = 0$. What is the ground space's energy and what is its degeneracy?

b. (2 points) Calculate the projector to the instantaneous ground space for general values of s with $\mu_b(s) = \cos(s)\mu$, $\mu_r(0) = \sin(s)\mu$ and $\mu_l(0) = 0$. Hint: *One brute force possibility is to use the many-body basis* $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$*, diagonalize the Hamiltonian, and find the projectors. Another one is to diagonalize the Hamiltonian in Majorana basis, and work out the projector in second quantization.*

c. (4^* points) [1] Solve Kato's adiabatic evolution for $\mu_b(s) = \cos(s)\mu$, $\mu_r(0) = \sin(s)\mu$ and $\mu_l(0) = 0$ for s from 0 to $\pi/2$.

$$K'(s) = P'(s)K(s). \tag{3}$$

d. (6 points) Combine the adiabatic operators of the following three-step evolution to get the "braiding operator", i.e., $K(3\pi/2)$, for

[1] * The asterisk marks extra hard exercises.

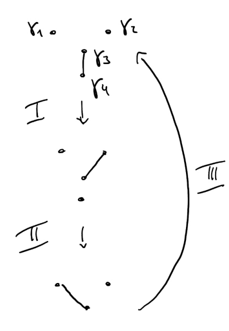
1. $\mu_b(s) = \cos(s)\mu$, $\mu_r(0) = \sin(s)\mu$ and $\mu_l(0) = 0$, for $s \in [0, \pi/2]$.
2. $\mu_b(s) = 0$, $\mu_r(0) = \sin(s)\mu$ and $\mu_l(0) = -\cos(s)\mu$ for $s \in [\pi/2, \pi]$.
3. $\mu_b(s) = -\sin(s)\mu$, $\mu_r(0) = 0$ and $\mu_l(0) = -\cos(s)\mu$ for $s \in [\pi, 3/2\pi]$.

Can you visualize the process? What is its physical meaning? Think, e.g., about the Majoranas being at the end of Kitaev Majorana chains (whose bulk is not considered here because it is gapped from the low-energy theory).

e. (3* points) [2] What does the braiding operator correspond to when expressed in the many-body basis $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$? Is it an Abelian or non-Abelian operator? Extra question: What if two additional Majorana modes are added that do not participate in the Hamiltonian?

Solution:
Given Majorana modes $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ and the Hamiltonian $H(s)$. To compute the ground state, we use the spinor γ

The periodic adiabatic evolution is:



$$\mathcal{H}(t) = \mathrm{i}\ [\mu_b(t)\gamma_3\gamma_4 + \mu_r(t)\gamma_2\gamma_3 + \mu_l(t)\gamma_1\gamma_3]$$

$$\gamma := (\gamma_1, \gamma_2, \gamma_3, \gamma_4)^T$$

$$= \frac{i}{2}\gamma^T \begin{pmatrix} 0 & 0 & \mu_l & 0 \\ 0 & 0 & \mu_r & 0 \\ -\mu_l & -\mu_r & 0 & \mu_b \\ 0 & 0 & -\mu_b & 0 \end{pmatrix} \gamma\ .$$

We introduce mixed Majoranas,

$$\tilde{\gamma} := R(s')\gamma = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(s') & \sin(s') & \\ 0 & 0 & 1 & 0 \\ 0 & \sin(s') & 0 & \cos(s') \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{pmatrix},$$

where R is a $SO(3)$ rotation around γ_3, exchanging γ_4 and γ_2. With trigonometric identities, we find

$$H(t) = \frac{i}{2} \begin{pmatrix} \gamma_1 \\ \cos(s)\gamma_2 + \sin(s)\gamma_4 \\ \gamma_3 \\ -\sin(s)\gamma_3 + \cos(s)\gamma_4 \end{pmatrix}^T \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & -\mu & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \cos(s)\gamma_2 + \sin(s)\gamma_4 \\ \gamma_3 \\ -\sin(s)\gamma_3 + \cos(s)\gamma_4 \end{pmatrix}$$

This is a fermionic Hamiltonian with the normal (diagonalizing) fermions $\tilde{c}_1 = \frac{1}{\sqrt{2}}(\tilde{\gamma}_1 - i\tilde{\gamma}_4)$ and

[2] *: The asterisk marks volunatary exercises.

$\tilde{c}_2 = \frac{1}{\sqrt{2}}(\tilde{\gamma}_2 - i\tilde{\gamma}_3)$. The mode c_1 has vanishing energy, seen by

$$\frac{i}{2}\tilde{\gamma}^T\tilde{h}\tilde{\gamma} = -\mu c_2^\dagger c_2 \ .$$

The ground space is the subspace with one c_2 particle. The projector is

$$\begin{aligned} P &= c_2^\dagger c_2 \\ &= \frac{1}{2}(\tilde{\gamma}_2 + i\tilde{\gamma}_3)(\tilde{\gamma}_2 - i\tilde{\gamma}_3) \\ &= 1 + i\tilde{\gamma}_3\tilde{\gamma}_2 \\ &\underline{\underline{= 1 + i\gamma_3(\cos(s)\gamma_2 + \sin(s)\gamma_4)}} \end{aligned}$$

To find the adiabatic evolution, we need to solve Kato's equation

$$\begin{aligned} K'(s) &= [P'(s), P(s)]K(s) \\ &\quad \Big| \quad P'(s) = i\gamma_3(\cos(s)\gamma_4 - \sin(s)\gamma_2) \\ &= -\{\cos^2(s)[\gamma_3\gamma_2, \gamma_3\gamma_4] - \sin^2(s)[\gamma_3\gamma_4, \gamma_3\gamma_2]\}K(s) \\ &\quad \Big| \quad [\gamma_3\gamma_2, \gamma_3\gamma_4] = -[\gamma_3\gamma_2, \gamma_3\gamma_4] \\ &\underline{\underline{= \gamma_2\gamma_4 K(s)}} \end{aligned}$$

Check prefactor

We solve Kato's equation as a typical first order differential equation and find

$$K(s) = e^{\gamma_2\gamma_4 s}P(0) \ .$$

We can see that $(2i\gamma_2\gamma_4)^2 = 1$, such that

$$\begin{aligned} e^{\gamma_2\gamma_4 s} &= e^{-i\frac{s}{2}2i\gamma_2\gamma_4} \\ &= \sum_{n=0}^{\infty}\frac{(2i\gamma_2\gamma_4)^n}{n!}(-i\frac{s}{2})^n \\ &= \sum_{\text{n even}}\frac{1}{n!}(-i\frac{s}{2})^n + 2i\gamma_2\gamma_4\sum_{\text{n odd}}\frac{1}{n!}(-i\frac{s}{2})^n \\ &= \cos\left(\frac{s}{2}\right) - 2i\gamma_2\gamma_4\sin\left(\frac{s}{2}\right) \ . \end{aligned}$$

Consequently, the propagator is

$$K(s) = (\cos\left(\frac{s}{2}\right) - 2i\gamma_2\gamma_4\sin\left(\frac{s}{2}\right))P(0) \ .$$

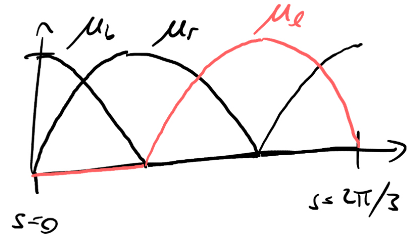
More generally, let $K(S_I, S_E)$ be the Kato propagator from adiabatic time S_I to S_E, then we get

$$K(S_I, s_E) = (\cos\left(\frac{s_E - S_I}{2}\right) - 2i\gamma_2\gamma_4 \sin\left(\frac{s_E - S_I}{2}\right))P(S_I) \ .$$

Following the evolution from $s = 0$ to $s = \frac{\pi}{2}$, we reach

$$\underline{\underline{K(0, \frac{\pi}{2}) = 2i\gamma_2\gamma_4 P(0)}} \ .$$

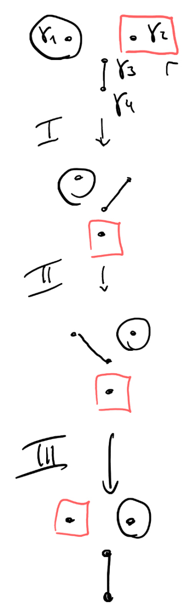
Let us now regard the time evolution



$$\mu_b(s) = \begin{cases} \cos(s) & 0 \leq s \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leq s \leq \pi \\ -\cos(s) & \pi \leq s \leq \frac{3\pi}{2} \end{cases}$$

$$\mu_r(s) = \begin{cases} \sin(s) & 0 \leq s \leq \frac{\pi}{2} \\ \sin(s) & \frac{\pi}{2} \leq s \leq \pi \\ 0 & \pi \leq s \leq \frac{3\pi}{2} \end{cases}$$

$$\mu_l(s) = \begin{cases} 0 & 0 \leq s \leq \frac{\pi}{2} \\ -\cos(s) & \frac{\pi}{2} \leq s \leq \pi \\ -\cos(s) & \pi \leq s \leq \frac{3\pi}{2} \end{cases}$$



The intuition is that the Majoranas get exchanged clockwise, reaching the initial Hamiltonian again. We observe that the steps one and two are obtained from the first one by swapping indices in $H(s)$ and adding the respective adiabatic time
$\mathrm{II} : s \to s + \frac{\pi}{2}, \ \ \mathrm{III} : s \to s + \pi.$

This means for $0 \leq s \leq \frac{\pi}{2}$ that

$$H(s) = H(s) \qquad \Rightarrow \ K(0, \frac{\pi}{2}) = -2i\gamma_2\gamma_4 P(0)$$

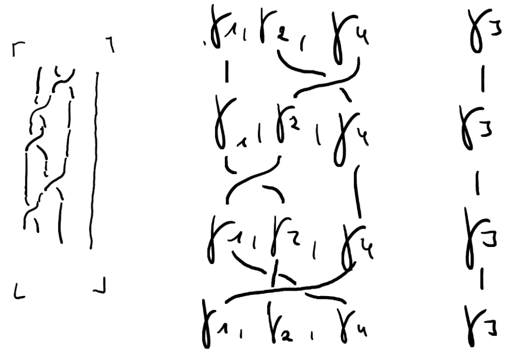
$$H(s + \frac{\pi}{2}) = H(s) \text{ with } \gamma_2 \to -\gamma_4 \ \& \ \gamma_4 \to \gamma_2 \qquad \Rightarrow \ K(\frac{\pi}{2}) = 2i\gamma_4\gamma_2 P(\frac{\pi}{2})$$

$$H(s + \pi) = H(s + \frac{\pi}{2}) \text{ with } \gamma_1 \to -\gamma_2 \ \& \ \gamma_2 \to \gamma_1$$

$$H(s + \pi) = H(s) \text{ with} \gamma_1 \to \gamma_4 \ \& \ \gamma_2 \to \gamma_1 \ \& \ \gamma_4 \to \gamma_2 \qquad \Rightarrow \ K(\pi, \frac{3\pi}{2}) = -2i\gamma_1\gamma_2 P(\pi)$$

Side remark: Braiding diagrams

We can depict this process intuitively by this diagram, which will play a role in the next lectures about anyons.



The crossing behind or in front on the left diagram comes from ± signs in the Hamiltonian.

For the total Kato propagator, we obtain

$$
\begin{aligned}
K(0, \frac{3\pi}{2}) &= K(\pi, \frac{3\pi}{2})P(\pi)K(\frac{\pi}{2}, \pi)P(\frac{\pi}{2})K(0, \frac{\pi}{2})P(0) \\
&= K(\pi, \frac{3\pi}{2})K(\frac{\pi}{2}, \pi)K(0, \frac{\pi}{2})P(0) \\
&= -2i\gamma_1\gamma_2 2i\gamma_4\gamma_2(-2i)\gamma_2\gamma_4 P(0) \\
&\underline{\underline{= -2i\gamma_1\gamma_2 P(0)}}
\end{aligned}
$$

The first part here is called the *braiding operator*

$$\boxed{B_{12} = -2i\gamma_1\gamma_2} \quad .$$

Remark on the nonabelian braiding of two free Majoranas

Note that the actual many body space $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ separates into "superselection sectors" with different parity, because the Hamiltonian & no observables mixes fermionic parities

$$H_A = \mathrm{span}\{|00\rangle, |11\rangle\}, \quad H_B = \mathrm{span}\{|01\rangle, |10\rangle\} \ .$$

Therefore, the ground space merely obtains a phase by the adiabatic evolution. This changes if we introduce a larger degeneracy with additional Majorana modes γ_5 & γ_6 that correspond to a zero mode (the Hamiltonian and hence the Kato propagator remains unchanged).

Similarly, we can say that

$$[B_{12}, B_{21}] = [B_{12}, -B_{12}] = 0$$

but $[B_{12}, B_{25}] \neq 0$, i.e., the group of braiding operators (proof that they form a group) is only non-Abelian if sufficiently many braiding operators exist.
Hence, we need at least four free Majoranas for adiabatic state manipulation and topological quantum computing.

End