

Topology in condensed matter physics

exercise sheet 1

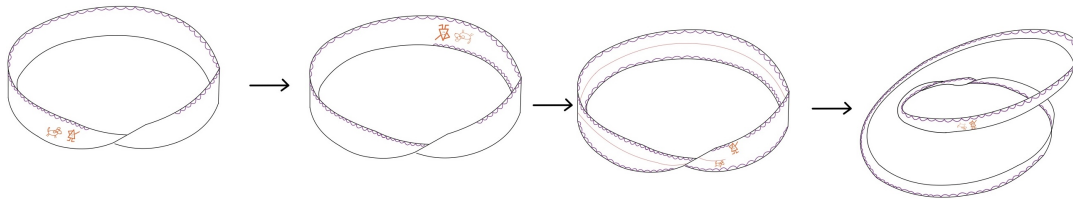
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1.1 Inspirations from topology

* This exercise is optional and for inspirational purpose.

1. (1* points) Watch Wind and Mr.Ug: <https://www.youtube.com/watch?v=Vmas4kr4Qyk>.
2. (2* points) Answer the question(s) that are posed at the end of the video.
3. (1* points) Extra question: If Wind finds her dog again after the earth quake, is the dog the same as before? If not, could there be serious problems?

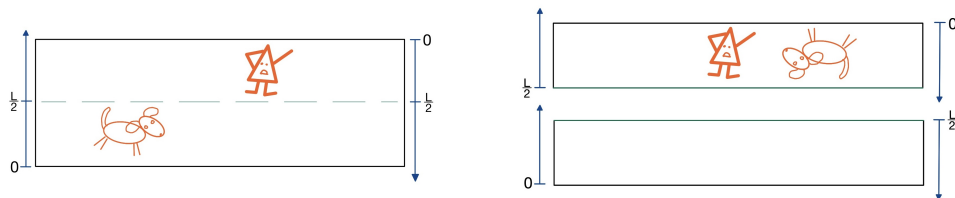
Solution:



When the Möbius strip is cut in half, instead of one half loop, there are two half loops. They are still connected, which means cutting a Möbius strip in half results in one long stripe with one full loop. In case Wind's dog is on the other side after the cut, it needs to complete one half loop to get to Wind.

The new strip is orientable. It now gets tricky. Since in our 3D world there are no Möbius-strips (because a Möbius-strip is 2D) our answer depends on interpreting the strip as 2D or 3D.

For a 3D strip, Wind's dog is for eternity captured on the other side of the strip. For a 2D strip, Wind's dog will be found by Wind again on the other side of her little world. But the dog is mirrored upside down.

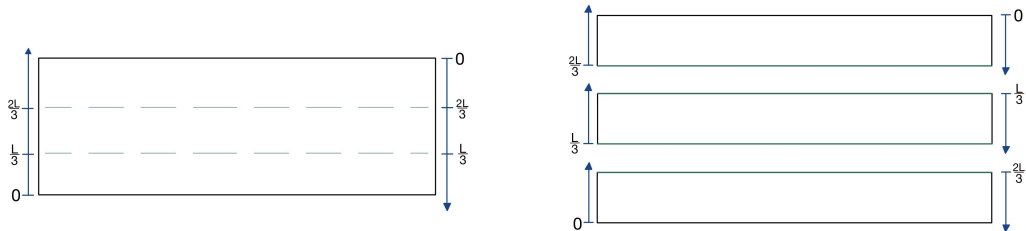


1.2 The Möbius strip

* This exercise is optional and for inspirational purpose.

- (2* points) Hypothesize what happens if you cut the Möbius strip parallel to its boundary, where the cut has a constant distance of $1/3$ of the width of the strip. Consequently, answer the question experimentally.
- (2* points) What happens if you cut the Möbius strip in an arbitrary (but constant) distance to its boundary?

Solution:



The lowest and highest piece in the gluing diagram are connected and are together twice as long as the middle stripe. We also see, that while the two stripes together have one full loop, the middle stripe has just a half loop and forms a Möbius stripe. When analysing the order of the cuts, we notice, that the Möbius stripe and the long stripe with one full loop are wind around each other.

For further discussion: Let n be the amount of cuts, l the half loops and L the full loops. We notice that

$$l = n + 1 \Rightarrow L = \frac{n}{2} + \frac{1}{2}.$$

For an even number of cuts $n = 2k, k \in \mathbb{N}$, there is always one Möbius strip.

1.3 Foundations of topology

- (2 points) Which of the following families are topologies of $X = \{1, 2, 3\}$?

(a) $\mathcal{T} = \{\{\}, \{1, 2, 3\}\}$

Solution:

Claim: $\mathcal{T} = \{\emptyset, \{1, 2, 3\}\}$ satisfies (O1) - (O3) and is therefore a topology, which is also called the *indiscrete- or trivial topology*.

We notice $\emptyset, X \in \mathcal{T}$, which proofs (O1). The intersection of both elements of \mathcal{T} is the empty set and $\emptyset \in \mathcal{T}$, which proofs (O2). With $\emptyset \cup \{1, 2, 3\} = \emptyset$, \mathcal{T} satisfies also (O3) and is therefore a topology. \square

(b) $\mathcal{T} = \{\{\{\}\}, 1, 2, 3\}$

Solution:

Claim: $\mathcal{T} = \{\{\emptyset\}, 1, 2, 3\}$ does not satisfy (O1) and (O3) and is therefore no topology.

It is sufficient to notice, the empty set is not an element $\emptyset \notin \mathcal{T}$. Therefore, \mathcal{T} does not satisfy (O1). This show that \mathcal{T} is not a topology. \square

(c) $\mathcal{T} = \{\{\}, 1, 2, 3, \{1, 2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

Solution:

Claim: $\mathcal{T} = \{\emptyset, 1, 2, 3, \{1, 2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ contains elements that are not sets. Hence, this cannot be a topology. If we interpret those singletons as sets, the want-to-be topology satisfies (O1) but not (O2) and (O3) and is therefore not a topology.

We directly see (O1) is true. However,

$$\{1, 2\} \cap \{1, 3\} = \{1\} \notin \mathcal{T},$$

which disproves (O2) and is therefore not a topology. \square

(d) $\mathcal{T} = \{\{2\}, \{1, 3\}, \{1, 2, 3\}, \{\}\}$

Solution:

Claim: $\mathcal{T} = \{\{2\}, \{1, 3\}, \{1, 2, 3\}, \emptyset\}$ satisfies (O1) - (O3) and is therefore a topology.

We directly see (O1) is true. For (O2), we show

$$\{2\} \cap \{1, 3\} = \{\} \in \mathcal{T}$$

$$\{2\} \cap \{1, 2, 3\} = \{2\} \in \mathcal{T}$$

$$\{1, 3\} \cap \{1, 2, 3\} = \{1, 3\} \in \mathcal{T}$$

and for (O3)

$$\{2\} \cup \{1, 3\} = \{1, 2, 3\} \in \mathcal{T}$$

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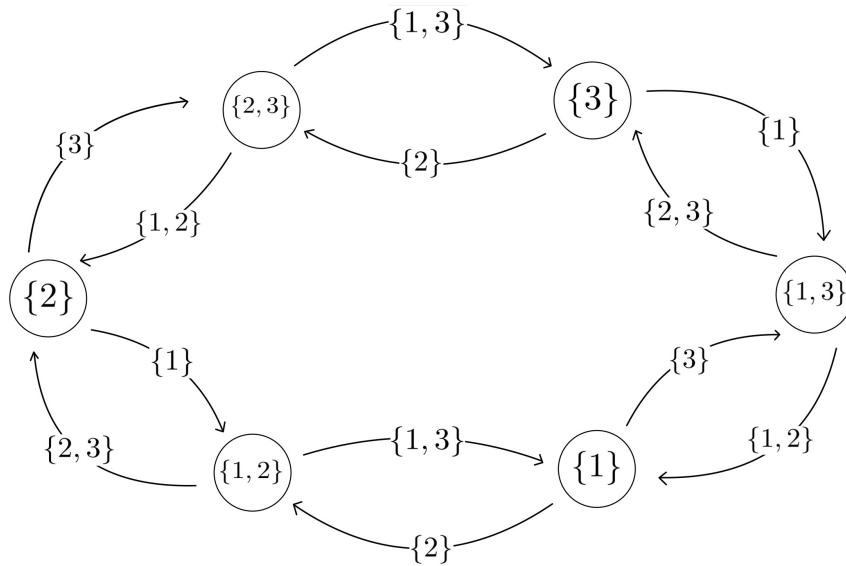
which proves \mathcal{T} is a topology. \square

2. (3 points) What are the remaining possible topologies of $X = \{1, 2, 3\}$?
Hint: Hard exercise. Listing this systematically requires some thinking.

Solution: To find the rest make a table of unions and intersections.

U/\cap	X	{2,3}	{1,2}	{1,3}	{1}	{2}	{3}	\emptyset
X	-	-	-	-	-	-	-	-
{2,3}	-	-	X/{2}	X/{3}	X/ \emptyset	{2,3}/{2}	{2,3}/{3}	-
{1,2}	-	X/{2}	-	X/{1}	{1,2}/{1}	{1,2}/{2}	X/ \emptyset	-
{1,3}	-	X/{3}	X/{1}	-	{1,3}/{1}	X/ \emptyset	{1,3}/{3}	-
{1}	-	X/ \emptyset	{1,2}/{1}	{1,3}/{1}	-	{1,2}/ \emptyset	{1,3}/ \emptyset	-
{2}	-	{2,3}/{2}	{1,2}/{2}	X/ \emptyset	{1,2}/ \emptyset	-	{2,3}/ \emptyset	-
{3}	-	{2,3}/{3}	X/ \emptyset	{1,3}/{3}	{1,3}/ \emptyset	{2,3}/ \emptyset	-	-

From the table, we generate a directed graph that helps us to construct the coarsest topology containing a present Family \mathcal{F} of subsets with the following algorithm. For every node: Check if the label is in \mathcal{F} . If yes, Follow every edge that is labeled by an element of \mathcal{F} and add the node you arrive at to \mathcal{F} . Repeat the algorithm until \mathcal{F} does not change anymore.



We observe the following: New elements of \mathcal{F} are generated if and only if an element not in \mathcal{F} lies in between two elements of \mathcal{F} . We can therefore relatively easy go through all the possibilities.

- 1x : $\mathcal{T} = \{\emptyset, X\}$
- 1x : $\mathcal{T} = \{\emptyset, X, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}\}$
- 6x : $\mathcal{T} = \{\emptyset, X, U\}, U \subset X$
- 3x : $\mathcal{T} = \{\emptyset, X, U, U^c\}, U \subset X$
- 6x : $\mathcal{T} = \{\emptyset, X, \{a\}, \{a,b\}\}, a, b \in X$ or $\mathcal{T} = \{\emptyset, X, \{a\}, \{b,c\}\}$
- 6x : $\mathcal{T} = \{\emptyset, X, \{a\}, \{a,b\}, \{a,c\}\}$ or $\{\emptyset, X, \{a\}, \{a,b\}, \{b\}\}$
- 6x : $\mathcal{T} = \{\emptyset, X, \{a\}, \{a,b\}, \{b\}, \{c,b\}\}$

\Rightarrow 29 topologies. 9 distinct types. Note that there could in principle be $|2^{2^x}| = 2^{2^3} = 256$ or (excluding X and \emptyset) $2^6 = 64$ sets of subsets to check if the set is a topology. It may be exhausting to check all of those by brute force.

End