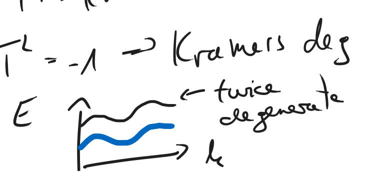
Lecture VII

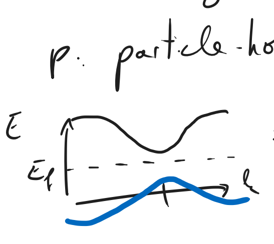
Hilbert space fibre bundle
tenfold way + examples

previous lecture — T, P, C symmetries & properties

T: time reversal
if $T^2 = -1 \Rightarrow$ Kramers deg.



P: particle-hole

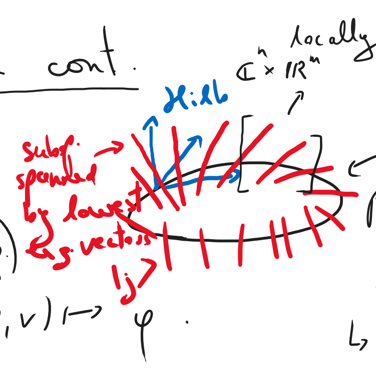


C: chiral symmetry

$\mathcal{H} = \begin{pmatrix} 0 & Q \\ Q^\dagger & 0 \end{pmatrix}$

Hilbert subspace fibre bundle cont.

repetition
$E \subseteq \mathcal{H}ilb \times B$ is the total space of fibre bundle (inherits topology as subspace of product sp.)
$E \xrightarrow{\pi} B$ with $\pi: E \to B$, $(\varphi, v) \mapsto \varphi$.



$\varphi \in B$
↳ For every $\varphi \in B$ there is a neighborhood $U \subseteq B$ with $U \approx \mathbb{R}^n$

$\varepsilon_\varphi, (|j\rangle_\varphi)_{j=1}^n$ defined by Hamiltonian $\mathcal{H}(\varphi)$ as eigenvalues & eigenvectors are continuous in φ.

π surjective: $\pi(E) = \{ \underbrace{\pi((\varphi, \underline{v}))} \mid \varphi \in B \} = \{ \varphi \mid \varphi \in B \} = B$ ✓

π continuous: By definition of the product space topology of $\mathcal{H}ilb \times B$.

local triviality: For $\varphi \in B$, take neighborhood $U \subseteq B$ with $U \approx \mathbb{R}^n$. Observe that $F(\varphi')$ with $\varphi' \in U$ is homeomorphic to $\mathbb{C}^n$ by

$\tilde{v} \in F(\varphi') \Rightarrow \tilde{v} = \sum_{j=1}^n \underbrace{\tilde{v}_j(\varphi)}_{\in \underline{\tilde{v}}(\varphi) \in \mathbb{C}^n} |j\rangle_\varphi$

$F(\varphi') = \pi^{-1}(\varphi)$

Then $i: U \times \mathbb{C}^n \to \pi^{-1}(U)$ with
$(\varphi, \underline{\tilde{v}}(\varphi)) \mapsto (\varphi, \sum_{j=1}^n \tilde{v}_j(\varphi) |j\rangle_\varphi)$ is
a homeomorphism with
$\pi(i(U \times \mathbb{C}^n)) = \pi(\{\varphi', \sum_{j=1}^n \tilde{v}_j(\varphi') |j\rangle_\varphi \mid \varphi' \in U\})$
$= \{\varphi' \mid \varphi' \in U\} = U$. ✓

Question: What happens if $\mathcal{H}(\varphi)$ discontinuous

Topological classification

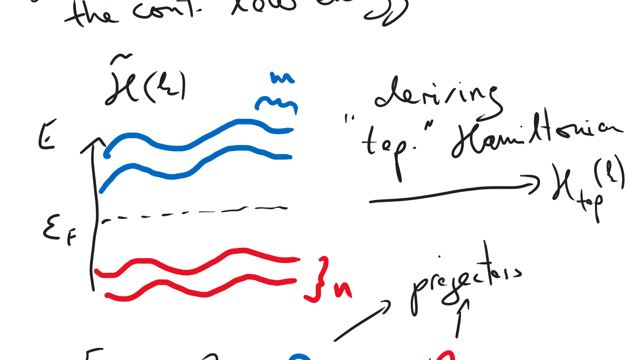
Topological classification differs for cont. theories & crystal (lattice) theories — Ryu, Schnyder Furusaki

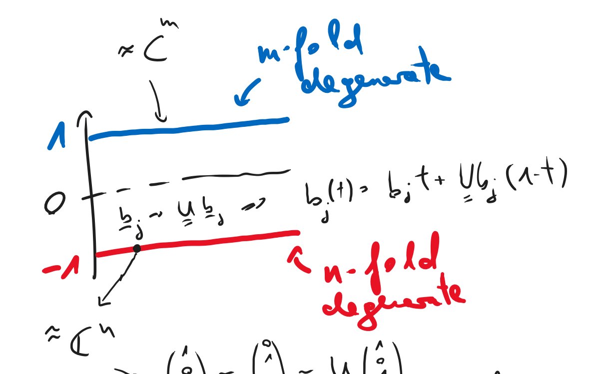
$\mathcal{H} = \int_{\mathbb{R}^d} d^d x \, \Psi^\dagger(x) \underline{\underline{\mathcal{H}}}(x) \Psi(x)$ + anomalous terms

$= \int d^d k \, \tilde{\Psi}^\dagger(k) \tilde{\mathcal{H}}(k) \tilde{\Psi}(k)$ + anomalous terms

$BZ = T^d$ (d-dimensional torus)
$S^d = \mathbb{R}^d / \infty$ [$\tilde{\Psi}(k) \to 0$ if $k \to \infty$ (low-energy theory) $\Rightarrow$ identify all points at $k = \infty$]

first focus on the cont. low-energy theories





$\to \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \sim U \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

↳ order/unitary transforms of only occ or only unocc. bands do not generate zero energy level crossing → factor out later

$\mathcal{H}_{top\,k} [= Q_k] = P_{occ} - P_{unocc}$ ← projectors

Q is unitary

$Q_k^\dagger Q_k = P_{occ}^\dagger - \cancel{P_{occ} P_{unocc}} - \cancel{P_{unocc} P_{occ}} + P_{unocc}$
$= P_{occ} + P_{unocc} = \mathbb{1} \Rightarrow Q \in U(n+m)$

Important equivalence classes for top. classification
. order of occ & unocc bands do not matter
$\Rightarrow$ equivalence classes $U(n+m) / \sim (U(m) \times U(n)) = \approx$
$\approx G(n+m, m)$.

$\Rightarrow$ Ultimately
$Q: S^d \to G(n+m, m)$.
(S^d: momentum space)

$\Rightarrow$ Topological classification of Hamiltonian by $\pi_d(G(n+m, m))$.
↳ target manifold G

recall
$\pi_1(X)$: homotopy cl. of $S^1 \to X$
$\pi_2(X)$: homotopy cl. of $S^2 \to X$
⋮
$\pi_d(X)$: homotopy cl. of $S^d \to X$

Only consider arbitrary large n & m to avoid special cases (unphysical)

$\pi_d(G)$

Cartan 1926-1927	P	T	C	d/1	2	3	4	
A	0	0	0	/	$\mathbb{Z}$	/	$\mathbb{Z}$	…
AIII	0	0	1	$\mathbb{Z}$	/	$\mathbb{Z}$	/	…
AI	1	0	0	/	/	/	$\mathbb{Z}$	
BDI	1	1	1	$\mathbb{Z}$	/	/	/	
D	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	/	/	
DIII	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
AII	-1	0	0	/	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
CII	-1	-1	1	$\mathbb{Z}$	/	$\mathbb{Z}_2$	$\mathbb{Z}_2$	
C	0	-1	0	/	$\mathbb{Z}$	/	$\mathbb{Z}_2$	
CI	1	-1	1	/	/	$\mathbb{Z}$	/	

A, AIII: 2-periodic Bott clock

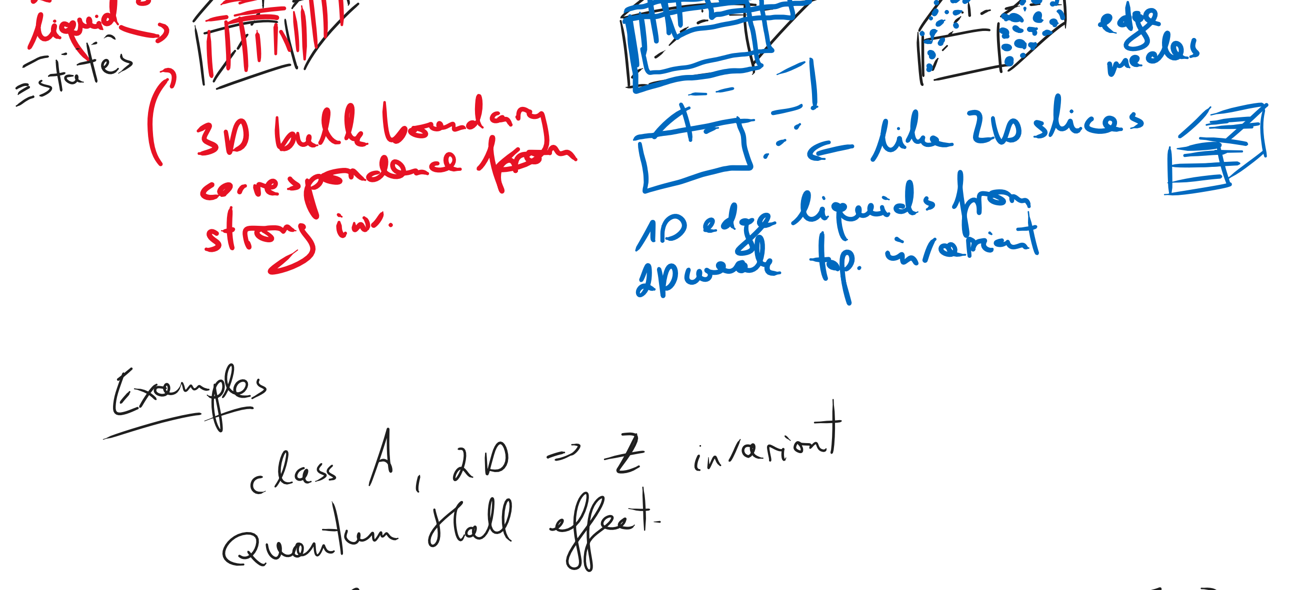
8-periodic Bott clock

for torus $(S^1)^d \to G$
→ find all eq. classes by $S^d \to G$, $S^{d-1} \to G$ …

○ : (low-energy) strong topological inv.
⬚ : weak topological invariants.

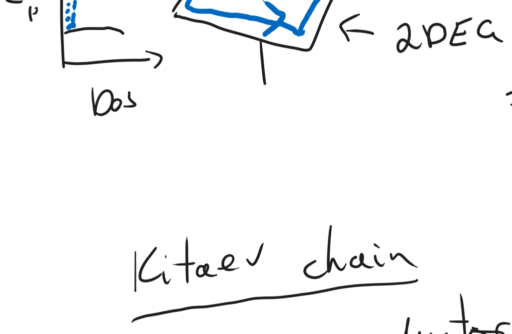
Remark: Additional P, T, C symmetries change the target manifold. [We do not start with all unitary top. Hamiltonians] $\Rightarrow$ different π_d $\Rightarrow$ different top. classification

Example: DIII material, top. index $\mathbb{Z}$, weak 1D & 2D top indices



Examples

class A, 2D $\Rightarrow \mathbb{Z}$ invariant
Quantum Hall effect.



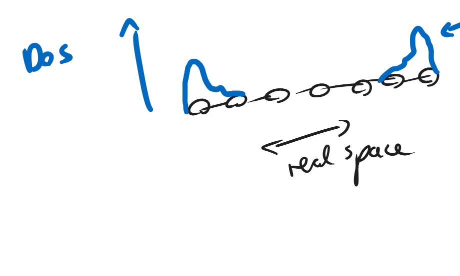
$\mathbb{Z} = \frac{1}{2\pi i} \int d^2k \, \mathrm{Tr}\{ P_k [\partial_{k_x} P_k, \partial_{k_y} P_k] P_k \}$

TKNN invariant 1982

$\mathbb{Z} \,\hat{=}$ number of edge modes.

Kitaev chain

1D top. superconductor in D with $\mathbb{Z}_2$ invariant



= Majora modes $\hat{=}$ nonabelian anyons

Example next time.