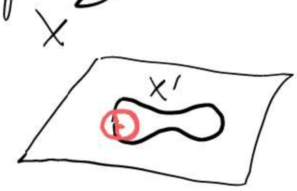
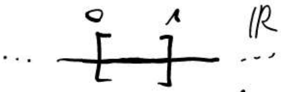
I.4. Constructing topological spaces

topological subspace

Given a topological space (X,τ) and a subset $X'\subseteq X$, then $\tau'=\{U\cap X' \mid U\in\tau\}$ is a topology of X' & (X',τ') is a <u>topological subspace</u>





proof

$X' = X\cap X'$ ✓

$\{\} = \{\}\cap X'$ ✓

$\bigcup\tau' = \bigcup(\tau\cap X') = (\bigcup\tau)\cap X'$ ✓

$\bigcap\tau' = \bigcap(\tau\cap X') = (\bigcap\tau)\cap X'$ ✓

direct sum of top. spaces

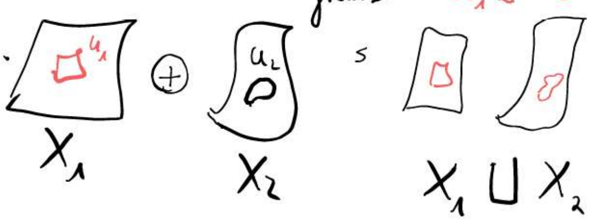
some index set

For topological spaces (X_i,τ_i), with $i\in I$, the space $(\bigsqcup_i X_i\ ,\ \tau=\{\bigsqcup_k U_k \mid U_k\in\tau_k\})$ is a topological space, the **direct sum** of (X_i,τ_i).

$\bigsqcup$ is the disjoint union:

$\{a,b\}\sqcup\{a,c\} = \{(1,a),(1,b),(2,a),(2,c)\}$

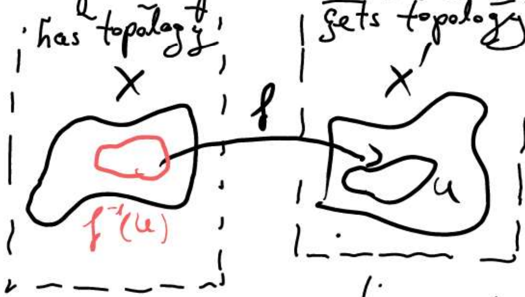
↳ from I



Final topology

For topological spaces (X_i, τ_i) and maps $f_i: X_i \to X'$, where X' is a set, we can define the finest topology on X' such that all f_i are continuous by

$$U' \in \tau \text{ if \& only if } f_i^{-1}(U) \text{ is open for all } i.$$



This is the final topology of (X', τ) induced by f_i.

f becomes continuous with the final topology.

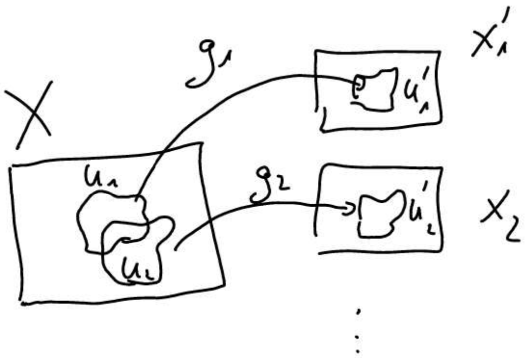
Needed for quotient spaces, e.g..

Induce topology for image set ("final")

Note: There are coarser topologies such that f_i are continuos, like the trivial topology (the preimages of "less" sets is still open). But we cannot make it finer. Then some of the new open sets in X' do not have an open preimage in X_i under f_i.

Initial topology

For a set X & maps $g_i : X \to X_i'$, with top. spaces X_i, the <u>initial topology</u> of X is the coarsest topology of X such that each g_i is continuous



Induces topology for base set
$\Rightarrow$ "initial" topology

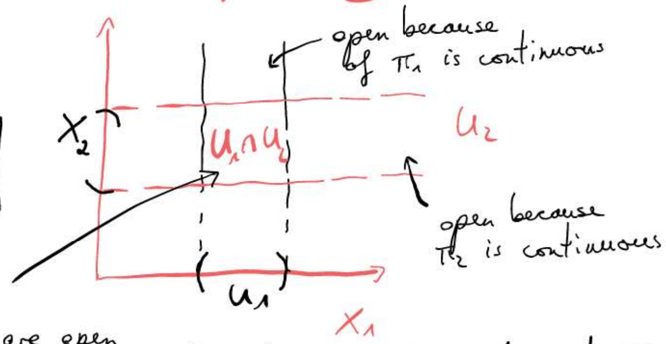
Note: We can make the topology finer by including more open sets in X but remain g_i continuous. E.g., the discrete topology (all sets are open).
We cannot make it coarser though. This would give an open set in some X_i whose preimage is not open.

direct product of top. spaces

For topological spaces (X_i, τ_i), consider the product set $X = \bigotimes_i X_i$. We then have projections $\pi_i : X \to X_i$ back to the factors of X, $\pi_i((x_1, x_2, \dots)) = x_i$. The product topology of X is the initial topology of these π_i,

i.e., the coarsest topology such that all π_i are continuous.

[also called Tychonoff topology]



open because finite intersections are open

→ problems with naïve definition here for index set I infinite.

Attention: for $i \in I$ with $|I|$ infinite, this definition is still good

Note: Tychonoff resolved the problem with infinite dimensions in the naive definition

$$\tau_{\text{naive}} = \left\{ \bigotimes_{U_j \in X_j} U_j \;\middle|\; U_j \text{ open} \right\}.$$

↳ this would include infinite intersections as well, not good. Constrain to finitely many.

The product topology is therefore also called Tychonoff topology.