Lecture 13 — Particle on a ring II (The story behind Mogami quantum computer)

Lecture 12 — Path integral quantization on non-simply connected configuration space

probability amplitude

$$K(x,t_x;y,t_y) = \int_{q(t_y)=y}^{q(t_x)=x} \mathcal{D}q\; e^{\frac{i}{\hbar}S(q)}$$

non-simply connected

$$\sum_{\gamma\in\pi_1(e)} \int_{q(t_y)=y}^{q(t_x)=x} \mathcal{D}q\; e^{\frac{i}{\hbar}S(q)}$$

γ is a projective unitary repres of $\pi_1(e)$

$$\gamma(g)\gamma(g') = e^{i\varphi}\gamma(gg') \quad \text{unitary}$$

$$\gamma(g)\gamma^\dagger(g) = \mathbb{1}, \quad \gamma(g)\gamma^\dagger(g) = \mathbb{1}$$

in case of 1D rep

Standard path $q_0:[t_0,t_x]\to e$

with $q_0(t_y)=y$, $q_0(t_x)=x$

$\Rightarrow$ loop $q\circ q_0^{-1}$ $\Rightarrow$ $[q\circ q_0^{-1}]$ well def.

! Physics invariant under choice of standard path

Later choose q', $q'(t_y)=y$, $q'(t_x)=x$

$$\Rightarrow K(x,t_x;y,t_y) = \sum_{\gamma\in\pi_1(e)} \gamma([q\circ q_0^{-1}]) \int_{[q\circ q_0^{-1}]=\gamma} \mathcal{D}q\; e^{\frac{i}{\hbar}S(q)}$$

$$\alpha\beta = \gamma \;\forall\, \alpha,\beta,\gamma\in G \text{ a group}$$

$$\Rightarrow \sum_{\beta\in G} f(\beta) = \sum_{\beta\in G} f(\alpha\beta) = \sum_{\beta\in G} f(\beta\alpha)$$

$$\Rightarrow K'(\dots) = \sum_{g\in\pi_1(e)} \gamma(g\circ[q_0\circ q_0'^{-1}]) \int_{\substack{[q\circ q_0'^{-1}]=g\\ q(t_y)=y,\ q(t_x)=x}} \mathcal{D}q\; e^{\frac{i}{\hbar}S(q)}$$

$\pi_1(e)$ is a group

unit group rep

$$e^{i\varphi}\gamma(g)\gamma([q\circ q_0'^{-1}]) = \gamma([q\circ q_0'^{-1}])$$

exercises!

$$[g]\cdot[\tilde g] = [g\cdot\tilde g]$$

$$[q\circ q_0'^{-1}]\cdot[q'\circ q_0'^{-1}] = [q\circ q_0'^{-1}]$$

$$K(x,t_x;y,t_y) = \sum_{g\in\pi_1(e)} \gamma(g) \int_{\substack{[q\circ q_0'^{-1}]=g\\ q(t_y)=y,\ q(t_x)=x}} \mathcal{D}q\; e^{\frac{i}{\hbar}S(q)}$$

$$e^{i\varphi}\gamma([q\circ q_0'^{-1}])$$

un-fixed constant

$$K(x,t_x;y,t_y)$$

$$\Rightarrow |K'(x,t_x;y,t_y)|^2 = |K(x,t_x;y,t_y)|^2 \underbrace{|e^{i\varphi}\gamma([q\circ q_0'^{-1}])|^2}_{=1}$$

Example of particle on a ring

$$\mathcal{H} = \frac{p^2}{2m}, \quad \mathcal{L} = \frac{1}{2m}\dot q^2$$

$$\gamma(g) = e^{i2\pi g}$$, with g is winding number of g

$$z_g = \frac{1}{L}\int_{t_y}^{t_x} \dot g(t)\,dt$$

$S^1 = [0,L]/_{0\sim L}$

$\pi_1(S^1) = \mathbb{Z}$

NOT:

$$\int \dot g\,dt = \int \frac{dg}{dt}\,dt$$

$$\int dg = g(t_x) - g(t_y)$$

Standard path:

$$q_0(t) = x\left(\frac{t_0-t}{t_0-t_x}\right)^{\in\mathbb{R}} + y\left(\frac{t_0-t}{t_0-t_y}\right)^{\in\mathbb{R}}$$

$$q_0(t_x) = y \;;\; q(t_x) = x \;\checkmark$$

$$\Rightarrow \text{probability amplitude}$$

$$K(x,t_x;y,t_y) = \sum_{g\in\pi_1(S^1)} \gamma(g) \int_{q(t_y)=y,\ q(t_x)=x} \mathcal{D}q\; e^{\frac{i}{\hbar}S(q)}$$

$$S(q) = \int_{t_y}^{t_x} \mathcal{L}(q)\,dt$$

$$[q\circ q_0'^{-1}] = g$$

absorbs

$$= \sum_{g\in\pi_1(S^1)} e^{i2\pi g} \int_{\dots} e^{\frac{i}{\hbar}\int_{t_y}^{t_x}\mathcal{L}(q)\,dt}$$

$$e^{i2\pi g} = e^{\frac{i}{\hbar}\int_{t_y}^{t_x}\frac{2\pi\hbar}{L}\dot q\,dt} = e^{\frac{i}{\hbar}\int_{t_y}^{t_x}\frac{2\pi\hbar}{L}\dot q\,dt + \frac{i}{\hbar}\int_{t_y}^{t_x}\frac{2\pi\hbar}{L}\dot q_0\,dt}$$

$$\int_{t_y}^{t_x}\dot q_0\,dt = q_0(t_x) - q_0(t_y) = x - y$$

$$= e^{\frac{i}{\hbar}\int_{t_y}^{t_x}\frac{2\pi\hbar}{L}\dot q\,dt} \cdot e^{\frac{i}{\hbar}\frac{2\pi\hbar}{L}(x-y)}$$

$$= e^{\frac{i}{\hbar}\int_{t_y}^{t_x}\left(\frac{2\pi\hbar}{L}\dot q + \frac{1}{2}m\dot q^2\right)dt} \cdot e^{-\frac{i}{\hbar}\int_{t_y}^{t_x}\frac{1}{2}m\dot q^2\,dt} \cdot e^{\frac{i}{\hbar}\frac{2\pi\hbar}{L}(x-y)}$$

$$= e^{\frac{i}{\hbar}\int_{t_y}^{t_x}\left(\frac{2\pi\hbar}{L}\dot q + \frac{1}{2}m\dot q^2\right)dt} \cdot e^{-\frac{i}{\hbar}\int_{t_y}^{t_x}\frac{1}{2}m\dot q^2\,dt} \cdot e^{\frac{i}{\hbar}\int_{t_y}^{t_x}\frac{2\pi\hbar}{L}\dot q_0\,dt}$$

$$= e^{\frac{i}{\hbar}\int_{t_y}^{t_x}\left(\frac{2\pi\hbar}{L}(\dot q+\dot q_0) + \frac{1}{2}m\dot q^2\right)dt} \cdot e^{-\frac{i}{\hbar}\int_{t_y}^{t_x}\frac{1}{2}m\dot q^2\,dt}$$

$$= e^{\frac{i}{\hbar}\int_{t_y}^{t_x}\left(\frac{2\pi\hbar}{L}\dot q + \frac{1}{2}m\dot q^2\right)dt} \cdot e^{-\frac{i}{\hbar}\int_{t_y}^{t_x}\frac{1}{2}m\dot q^2\,dt}$$

$$\int_{t_y}^{t_x}\dot q_0\,dt = q_0(t_x) - q_0(t_y) = x - y$$

$$= e^{\frac{i}{\hbar}\int_{t_y}^{t_x}\left(\frac{2\pi\hbar}{L}\dot q + \frac{1}{2}m\dot q^2\right)dt} \cdot e^{-\frac{i}{\hbar}\int_{t_y}^{t_x}\frac{1}{2}m\dot q^2\,dt} \cdot e^{\frac{i}{\hbar}\frac{2\pi\hbar}{L}(x-y)}$$

looks like the action

$$K(x,t_x;y,t_y) = e^{\frac{i}{\hbar}\frac{2\pi\hbar}{L}(x-y)} \sum_{g\in\pi_1(S^1)} \int_{\substack{q(t_y)=y\\ q(t_x)=x}} \mathcal{D}q\; e^{\frac{i}{\hbar}\int_{t_y}^{t_x}\left(\frac{2\pi\hbar}{L}\dot q + \frac{1}{2}m\dot q^2\right)dt}$$

$$= e^{\frac{i}{\hbar}\frac{2\pi\hbar}{L}(x-y)} \int_{\substack{q(t_y)=y\\ q(t_x)=x}} \mathcal{D}q\; e^{\frac{i}{\hbar}S'(q)} \quad \text{with } S'(q) = \int_{t_y}^{t_x}\mathcal{L}'\,dt$$

$$\mathcal{L}'(q) = \mathcal{L}(q) + \frac{2\pi\hbar}{L}\dot q$$

$$p = \frac{\partial\mathcal{L}'}{\partial\dot q} = m\dot q + \frac{2\pi\hbar}{L}$$

In order to interpret this transform to Hamiltonian mechanics

$$\Rightarrow \mathcal{H}' = \mathcal{L}' - p\dot q = -\frac{1}{2}m\dot q^2 = \frac{1}{2m}\left(p^2 + \frac{4\pi^2\hbar^2}{L^2} - p\frac{2\pi\hbar}{L}\right)$$

$$\dot q = \frac{1}{m}\left(p - \frac{2\pi\hbar}{L}\right) \quad (!)$$

$$\mathcal{H} = \frac{1}{2m}\left(-\frac{\hbar}{i}\left(p\cdot\frac{\partial}{\partial q} - \frac{2\pi\hbar}{L}\right)\right)^2 = \frac{\left(p\cdot\frac{\hbar}{i}\left(\frac{\partial}{\partial q} + p\right)\right)^2}{2m}$$

$$= \frac{\left(p\cdot\frac{\partial}{\partial q}\right)^2}{2m}$$

observation shifted momentum !

Hamiltonian is equivalent to normal particle on a ring + magnet potential

Aharonov Bohm effect

e.g. $\vec A(x,y,z) = \frac{\Phi}{2\pi\sqrt{x^2+y^2}}\vec e_\varphi$

$$\Rightarrow B(x,y,z) = \nabla\times\vec A \propto \delta(\sqrt{x^2+y^2})\vec e_z$$

$\Rightarrow$ minimal coupling $p \to p + e\vec A \vec e_\varphi = p\cdot\frac{\Phi}{2\pi g}$

$$\to p + \frac{\Phi}{L}.$$

$\Rightarrow$ Hamiltonian

$$\mathcal{H}_{AB} = \frac{\left(p + \frac{e\Phi}{L}\right)^2}{2m}$$

compare with our

$$\mathcal{H} = \frac{\left(p + \frac{2\pi\hbar}{L}\right)^2}{2m}$$, we see that $2\pi\cdot\frac{e\Phi}{h}$

matches Aharonov - Bohm effect

1) Path integral quantization includes elegant derivation of minimal coupling

2) Aharonov Bohm effected is inferred from particle on a ring gives amplitude without knowing anything what happens outside of ring.

Wikipedia 1391