Lecture 12 Quantization of not simply connected spaces & particle on a ring

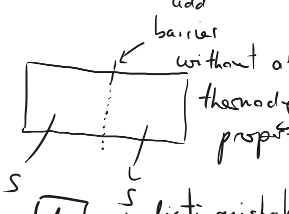
Lecture 11

$$\begin{tikzcd} \boxed{d/c} \arrow[r] \arrow[d] & \boxed{d/q} \arrow[d, "\text{symm./as}"] \\ \boxed{i/c} \arrow[r, "\text{qu.}"] & \boxed{i/q} \end{tikzcd}$$

Lec. 11 ←

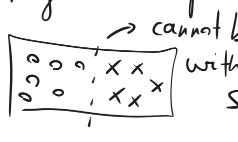
<u>Comment</u>: entropy

$S(\alpha N, \alpha V, \alpha U) = \alpha S(N, V, U)$

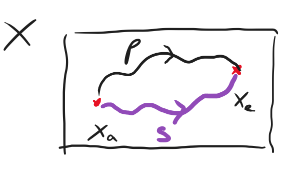


⇒ only possible if factor $\boxed{\frac{1}{N!}}$ → indistinguishable is included in partion function (indistinguishable part.)

Gibb's paradox (exercises 2) ⇒ Sackur-Tetrode equation



$\underline{\text{II}}$. 2.1. Extend assignment of paths to $\pi_1(X)$.



Introduce $s: X \times X \times [0,1] \to X$ standard path from each point in X to each other point in X.

Then we assign $[p \circ s^{-1}] \in \pi_1(X)$ to p & thereby have extended the notion of π_1.

$\underline{\text{III}}$ 2.2. Path integral quantization

Given a system of (distinguishable) particles in $\mathbb{R}^d$ & its classical $S = \int_{t_a}^{t_e} dt\, \mathcal{L}$, then Feynman's path integral quantization is:

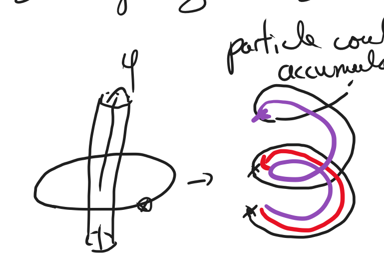
→ The probability that the particles are measured in conf. $\underline{x} = (x_1, \dots, x_n)$ at a time t_e when having been in configuration $\underline{y} = (y_1, \dots, y_n)$ at time t_a is abs. square of the probability amplitude $K(\underline{x}, t_e; \underline{y}, t_a)$ $\left(= \langle \underline{x}, t_e | U_{t_a, t_e} | \underline{y}, t_a \rangle\right)$.

- The prob. amplitude is

$$K(\underline{x}, t_e; \underline{y}, t_a) = \int\limits_{\substack{p(t_a) = \underline{y} \\ p(t_e) = \underline{x}}} Dp \; e^{i\hbar S(p)},$$

where p are all possible paths with boundary condition $p(t_a) = \underline{y}$; $p(t_e) = \underline{x}$ in configuration space, and $S(p) = \int_{t_a}^{t_e} \mathcal{L}(p)\, dt$ is the action of a path.

Idea for generalisation: Allow different weights/phases for different homotopy classes of paths



→ Dowker 1972
→ Schulman 1968
Laidlaw & DeWitt 1971.

$\underline{\text{III}}$ 2.3. quantization of non-simply connected spaces

Generalized propagator

$$K(\underline{x}, t_e; \underline{y}, t_a) = \sum_{g \in \pi_1} \Big[\int \underbrace{D_g p}_{\text{only } [p] = g} e^{iS(p)} \Big] \gamma(g)$$

with

$\gamma: \pi_1(X) \to \underline{\mathbb{C}}$ having properties

↳ possible to extend to $\mathbb{C}^{n \times n}$ ⇒ non-abelian anyons

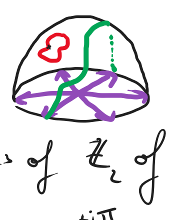
- $\gamma(g \cdot g') = \gamma(g) \cdot \gamma(g') \cdot e^{i\varphi}$ with a constant phase (indep. of $g, g' \in \pi_1(X)$).

⇒ γ is a projective representation of $\pi_1(X)$.

- $|\gamma(g)| = 1$ (unitary projective representation)

<u>exercises</u>
(indep. of choice of standard path)

$\underline{\text{III}}$ 3. Anyons, bosons, fermions

<u>3D</u> $\mathcal{C}_2^{3D} = \mathbb{R}^3 \times (0, \infty) \times$  , $\pi_1(\mathcal{C}_2^{3D}) = \mathbb{Z}_2$

⇒ "only two" representions of $\mathbb{Z}_2$ of form $\gamma: \pi_1 \to \mathbb{C}$,

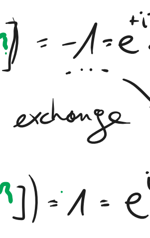
$\gamma_F([\cdot]) = 1$, $\gamma_F([\cdot]) = -1 = e^{+i\pi}$ ⟺ fermions

↗ contractible paths, no exchange

↳ exchange ↘ corr. complex phase

$\gamma_B([\cdot]) = 1$; $\gamma_B([\cdot]) = 1 = e^{i \cdot 0}$ ⟺ bosons

⇒ only <u>choice</u> are fermions & bosons (also in 4D, 5D, …)

<u>2D</u> $\mathcal{C}_2^{2D} = \mathbb{R}^2 \times$  ⇒ $\pi_1(\mathcal{C}_2^{2D}) = \mathbb{Z}$.

⇒ infinite number of projective unitary representations

$\gamma_\varphi([\cdot]) = e^{i\varphi z}$. With $\varphi \in [0, 2\pi)$ labeling different representations

↳ winding number z

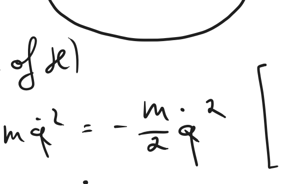
$\gamma_0(\dots) = 1$ ⟺ Bosons

$$\gamma_\pi(\dots) = \begin{cases} 1 & \text{if } z \text{ is even} \\ -1 & \text{if } z \text{ is odd} \end{cases} \iff \text{Fermions}$$

$\gamma_{\varphi \neq 0, \pi}(\dots) = e^{i\varphi z}$ ⟺ <u>anyons</u> (Frank Wilczek article)

Example: <u>free particle on a ring</u>

$S^1 = \mathbb{R}/(L \cdot \mathbb{Z})$



Hamiltonian $\mathcal{H} = \frac{p^2}{2m}$

Lagrangian (Legendre transform of $\mathcal{H}$)

$$\mathcal{L} = \mathcal{H} - p \cdot \dot{q} = \frac{m^2 \dot{q}^2}{2m} - m\dot{q}^2 = -\frac{m}{2}\dot{q}^2 \qquad [\mathcal{L} = -T + U]$$

with $\dot{q} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m} \iff p = m\dot{q}$.

Fundamental group of S^1: $\pi_1(S^1) = \mathbb{Z}$.

⇒ projective representations $\gamma_\varphi: \mathbb{Z} \to \mathbb{C}$, $\varphi \in [0, 2\pi)$;

$\gamma_\varphi(z) = e^{i\varphi z}$.

⇒ path integral

$$K(x, t_e; y, t_a) = \sum_{z \in \mathbb{Z}} \gamma_\varphi(z) \int\limits_{\substack{q(t_a) = y \\ q(t_e) = x}} Dq\; e^{-i\int_{t_a}^{t_e} \frac{m}{2}\dot{q}^2(t)\,dt}.$$

<u>Altland Simons book</u>

$$= \sum_{z \in \mathbb{Z}} e^{i\varphi z} \left(\frac{m}{2\pi i\hbar(t_e - t_a)}\right)^{\frac{1}{2}} e^{\frac{i}{\hbar}\frac{m}{2(t_e - t_a)}(q_e - q_a + zL)^2}\, \Theta(t_e - t_a).$$

↳ (can be calculated).