Lecture 11 — Anyons (aux Kaiserau & Myrheim)

Lecture 10: Kato's adiabatic theorem (proof)

$s = \frac{t}{T} \in [0,1]$ — start $t_0 = 0$, $t_1 = T$ — $\Rightarrow$ slow — $K(s) = [P_0(s), \dot{P}_0(s)]$

$K(0) \mapsto_{t_0} K(1) \mapsto_{t_1}$ — Properties on subspace — transportation — adiabatic — topological QC

Note: In some systems $K(s)$ is insensitive to changes (continuous, not change phase) of other in Hamiltonian, besides s.

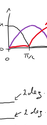
Example: Majoranas

$H(s) = \mu_1(s)\gamma_2\gamma_3 + \mu_2(s)\gamma_1\gamma_3 + \mu_3(s)\gamma_1\gamma_2$

$\mu_1(s) = \begin{cases} \cos(s) & 0 \le s \le \frac{\pi}{2} \quad \mathcal{I} \\ 0 & \frac{\pi}{2} \le s \le \pi \quad \mathcal{II} \\ -\sin(s) & \pi \le s \le \frac{3\pi}{2} \quad \mathcal{III} \end{cases}$

$\mu_2(s) = \begin{cases} \sin(s) & 0 \le s \le \pi/4 \quad \mathcal{I} \\ \sin(s) & \frac{\pi}{4} \le s \le \frac{\pi}{2} \quad \mathcal{II} \\ 0 & \pi \le s \le 3\pi/4 \quad \mathcal{III} \end{cases}$

$\mu_3(s) = \mu \begin{cases} 0 & \cdots \\ -\cos(s) & \cdots \\ -\cos(s) & \cdots \end{cases}$



Projected to ground space is

$P = (1 - c_1^\dagger c_1)$

$K(0, \frac{\pi}{2}) = -2i\,\gamma_1\gamma_3\,P(0)$ — to exchange/braiding operator between γ_1 & γ_3

$K(\frac{\pi}{2}, \pi) = 2i\,\gamma_1\gamma_2\,P(\frac{\pi}{2})$

$K(\pi, \frac{3\pi}{2}) = -2i\,\gamma_1\gamma_3\,P(\pi)$

$\Rightarrow K(0, \frac{3\pi}{2}) = K(\frac{\pi}{2}, \pi)\,K(0, \frac{\pi}{2}) = -2i\,\gamma_1\gamma_2\,P(0)$

$\Rightarrow$ corresponds to exchange/braiding of γ_1 & γ_2!

remark: independent of concrete dependence of $\mu_3(s)$ / $\mu_1(s)$ / $\mu_2(s)$, as long as ground state closed & intermediate steps $\mathcal{I}$, $\mathcal{II}$, $\mathcal{III}$ are realized

remark 2: Because of parity conservation (number of fermions remains odd or even, see general Hamilt. is $H = \sum_{k,m} t_{km} c_k^\dagger c_m - i\sum_{k,m} \Delta_{km} c_k^\dagger c_m^\dagger + h.c.$)

$\Rightarrow$ 2 degenerate ground eigenstates must be by $K(\frac{\pi}{2})$

$\Rightarrow$ becomes $\pm i$ (not bosons, not fermions)

To more states possible for larger degeneracy

More Majoranas $\{\gamma_1, \gamma_2, (\gamma_3, \gamma_4), (\gamma_5, \gamma_6), \gamma_7, \gamma_8\}$

replacement nontrivial braiding gate.

III Anyons (aux Kaiserau & Myrheim '77)

$\Leftrightarrow$ Goldhaber & the Wilczek '63
Schulman '68
Laidlaw '72

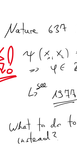
Nature 637 314 (2025)

$\psi(x_1, x_2) = e^{i\pi}\psi(x_1, x_2) = e^{2\cdot i}\psi(x_1, x_2)$ for exchangeable

$\Rightarrow \forall \in \mathbb{Z}\pi$

$\Leftrightarrow$ see 1378

What to do to describe all particle statistics instead?



III.1 Classical theory of indist particles

phase space $\mathbb{R}^{dN} \times \mathbb{R}^{dN}$ — real space — momentum space

problem: $\mathbb{R}^{dN}$ includes unimportants

hardcore particles/constraint $(x_j \neq x_k$ or $x_3 \neq x_4)$

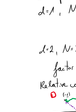
To construct configuration space of indist particles, form the configuration space of dist atoms

equivalence relation $(x_i)_{i=1}^N \sim (x_j)_{j=1}^N$ if there is permutation $\sigma(i) = j$

$\mathcal{C}_N^d = (\mathbb{R}^{Nd} \setminus D)/\sim$ — With some phase assumptions $D = \{x_j = x_k \text{ with } j \neq k\}$ such that $x_j \neq x_k$

Examples:

d=1, N=2



d=2, N=2

$\mathbb{R}^{2N} / \mathbb{R}^2$ — cannot discuss factor out center of mass

Relative coordinates

Three dimensions & two particles original configure space $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ (center of mass / invariant) $= \mathbb{R}^3 \times \mathbb{R}^3$ (6-dim) — S^2 relative motion

$\mathcal{C}_2^3$ is real projective space of dim 2

with 1D representation $\mathbb{Z} \to \mathbb{Z}_2 \to 1$ & multiple corresponds to bosons & fermions as group after quantization

remark: $\pi_1(\mathcal{C}_{2n}^{2n}) = \mathbb{Z}_2$ — only bosons & fermions for $d \geq 3$

additional statistics (many) exist in 2D

remark 2: $\pi_1(\mathcal{C}_N^2) = B_N$ — braid group on N strands.