Lecture 10 — Proof of the adiabatic theorem of quantum mechanics

Lecture 3 — Finished rather than ... Majorana zero modes ... Quantum computing, adiabatic theorem

III.2 Proof of the adiabatic theorem of quantum mechanics

Particular Setting: time-dependent Hamiltonian $\hat{H}(t)$ — eigenvalues gapped, eigenvectors gapped, eigenspaces gapped

Full Schrödinger Eq: $i\partial_t|\psi_t\rangle = \hat{H}(t)|\psi_t\rangle$ (S)

$|\psi(t)\rangle = U_t(t)|\psi(0)\rangle \; \forall\, |\psi(0)\rangle \in \mathcal{H}$

$\Rightarrow i\partial_t U_t(t) = \hat{H}(t) U_t(t) \;\forall t$ (Sch)

Solution: (introduce T (time ordering symbol))

definition product $T(A(t_1)B(t_2))$ $= \begin{cases} A(t_1)B(t_2) & t_1 \ge t_2 \\ B(t_2)A(t_1) & t_2 \ge t_1 \end{cases}$

& exponential formally $T(A+B) = e^{-\frac{1}{2}[A,B]} e^{A} e^{B}$ if $[A,[A,B]] = [B,[A,B]] = 0$

Then $U_t(t) = T\, e^{-i\int_0^t \hat{H}(t')dt'}$ ✓

proof: $-i\hat{H}(t)U_t(t)$ + $i\partial_t U_t(t) = T\, i\partial_t \left(e^{-i\int_0^t \hat{H}(t')dt'}\right) = T\, e^{-i\int_0^t \hat{H}(t')dt'} \left(-i\hat{H}(t)\right) = -i\hat{H}(t)U_t(t)$ ✓

but $U_t(t)$ is high-dimensional $\Rightarrow$ in general useful/relevant expression for U_t

Need to simplify assumptions: "infinitely slowly changing system"

$\hat{H}_\varepsilon(t) = \hat{H}\left(\frac{t}{\varepsilon}\right) = \hat{H}(s)$ independent of ε

By letting $\varepsilon \to 0$, we recall infinitely slow, i.e. adiabatic

History: 1928 Born & Fock — adiabatic evolution/phase invariant

then came → 1959 Kato — #/degeneracies

1984 Berry — generic/quantum subspace

adiabatic manipulation in QC

↳ Berry phase (non-degenerate)

↳ Wilczek-Zee (degenerate)

adiabatic theorem (Born):

If a system in a gapped eigenspace of the Hamiltonian is changed sufficiently slowly, the system remains in the corresponding instantaneous eigenspace.

proof: 1) Particular solution to collect useful properties

2) proof adiabatic theorem

3) Understand adiabatic evolution by parallel transp

reformulate in adiabatic time

	real time	adiabatic time
Parameter	t time	$s = t/\varepsilon$ adiab time
	t final time	τ final time
Hamil	$\hat{H}_\varepsilon(t)$	$\hat{H}(s) = \hat{H}_\varepsilon(\varepsilon s)$
unitary evo	$U_\varepsilon(t)$	$\tilde{U}_\varepsilon(s) = e^{i\int_0^{\varepsilon s}\lambda(t')dt'} U_\varepsilon(\varepsilon s) = e^{i\int_0^s \lambda(\varepsilon s')ds'} U_\varepsilon(\varepsilon s)$
eigenval of Hamil	$\frac{d\lambda}{dt} = \frac{d}{ds}\frac{ds}{dt}\lambda = \frac{1}{\varepsilon}\frac{d\lambda}{ds}$	
eigenvec	$\frac{d\psi}{dt}$	$\frac{d\psi}{ds} = \frac{d\psi}{dt}\frac{dt}{ds} = \frac{1}{\varepsilon}\frac{d\psi}{dt}$

reformulate Schrödinger in adiabatic time

$i\frac{d}{dt}U_\varepsilon(t) = i\frac{d}{dt}\tilde{U}_\varepsilon(s) = \hat{H}(s)\tilde{U}_\varepsilon(s)$

$i\frac{d}{ds}\tilde{U}_\varepsilon(s) = i\left(\lambda(s)\tau\left(e^{-i\int_0^s \lambda(s')ds'} U_\varepsilon(s)\right) + e^{-i\int_0^s \lambda(s')ds'} \tau \frac{d}{ds} U_\varepsilon(s)\right)$

$= T\left(\hat{H}(s) - \lambda(s)\right)\tilde{U}_\varepsilon(s)$ (SA)

$\Rightarrow i\tilde{U}_\varepsilon'(s) = -i\tau\left(\hat{H}(s) - \lambda(s)\right)\tilde{U}_\varepsilon(s)$

1) Particular solution (Kato)

The evolution in the adiab regime follows $K(s)$ with

(Kat) $K'(s) = \left[P'(s)P(s) - P(s)P'(s)\right]K(s)$ $\left[\frac{d}{ds}K(s) = K'(s)\right]$

$\lambda(s) \to$ [connection]

with $P(s)$ being the projection on instantaneous eigenspace with eigenval $\lambda(s)$.

properties of $K(s)$:

- $K(s) = T\, e^{\int \lambda(s)K(s)ds'} \cdot P(0)$ solution

Justified (Kat): $K'(s) = P'(s)P(s)K(s) = K(s)$ because $P(s) = P(s)$ [see exercises]

$\because P(s)P(s)K(s) = P(s)P(s)K(s) = P(s)K(s)$

(P) $P^2 = P$; $PP' + P'P = P'$; $PP'P + P'P^2 = P'P$; $\cdot PP' = 0$

2) proof of adiabatic theorem

Define $S(s) = \sum_{\mu(s)\in\lambda(s)} P_{\mu(s)}$ eigenvals of $\lambda(s)$

then $(\lambda(s) - \lambda(s))\,S(s) = \sum_{\mu\neq\lambda} (\mu(s) - \lambda(s))\, P_{\mu(s)}$

& $\lambda(s) = \sum_{\nu(s)\in\lambda(s)} \nu(s) P_{\nu(s)}$ & $P_{\nu(s)} P_{\mu(s)} = \delta_{\nu,\mu} P_{\mu(s)}$

(AS) $(\lambda(s) - \lambda(s))\,S(s) = \sum_{\mu\neq\lambda} \mu(s)(\mu(s) - \nu(s))\, P_{\mu(s)} = \mathbb{1} - P$

Consider (A) $\left[\tilde{U}_\varepsilon^\dagger(s)\, K(s)\right]' = \tilde{U}_\varepsilon^\dagger(s) K(s) + \tilde{U}_\varepsilon^\dagger(s) K'(s)$

$\overset{(A)/(SA)}{=} \tilde{U}_\varepsilon^\dagger(s) \cdot \frac{i}{\varepsilon}\tilde{U}_\varepsilon(s)\left(\lambda(s) - \lambda(s)\right) \overset{(AS)}{K(s)} + \tilde{U}_\varepsilon^\dagger(s) K'(s)$

$\approx \tilde{U}_\varepsilon^\dagger P' K(s)$

$= \tilde{U}_\varepsilon^\dagger(1) \, i\tau \left(\lambda(s) - \lambda(s)\right) P(s) K(s) + \xi$

$\approx \tilde{U}_\varepsilon^\dagger \left(\mathbb{1} - P\right) K'(s) + \tilde{U}_\varepsilon^\dagger P' K(s)$

$\left[\because P K'(s) \overset{(Kat)}{=} PP'K(s) = \frac{PP'P}{(P)=0} K(s)\right]$

$\overset{(AS)}{=} \tilde{U}_\varepsilon^\dagger \left(\lambda(s) - \lambda(s)\right) S(s) K'(s)$

$= -\frac{i}{\varepsilon}\tilde{U}_\varepsilon^\dagger S(s) K'(s)$ — Sec $\frac{d}{ds}$ adiabatic $\Rightarrow$ vanishes if $\nu \neq \mu$ or λ discr.

More convenient form: Integrate from 0 to s (A)

$\Rightarrow$ (B) $\tilde{U}_\varepsilon^\dagger(s) K(s) - \tilde{U}_\varepsilon^\dagger(0) K(0)$

B.1) $= \frac{i}{\varepsilon}\int_0^s \tilde{U}_\varepsilon^\dagger(s') S(s') K'(s') ds'$

$S(s), K(s)$ ind of τ

& $\tilde{U}_\varepsilon^\dagger(s) U_\varepsilon(s) = \mathbb{1}$ — unitary matrices cannot compensate for divergence

$\| U_\varepsilon \| < \infty$

$\Rightarrow \tilde{U}_\varepsilon(s)\tilde{U}_\varepsilon^\dagger(s) \cdot \tilde{U}_\varepsilon(s)$ (natural)

$\Rightarrow \boxed{K(s) = \tilde{U}_\varepsilon(s) P(0) + \frac{i}{\varepsilon}\tilde{U}_\varepsilon(s)\int_0^s \tilde{U}_\varepsilon^\dagger(s') S(s') K'(s') ds'}$ □

Geometric meaning of the adiabatic evolution

Consider $K'(s) = P'(s) K(s)$, ala Euler iteration

$K(s+\delta) = K(s) + \delta K'(s)$ $\quad \delta K(s) = \delta P'(s) K(s) = (P(s) + \delta P'(s)) K(s)$

to infinitesimally small $\Rightarrow K(s+\delta) = P(s+\delta) K(s)$

$\Rightarrow$ Solution of Kato's equation is obtained by step-by-step iteration

$K(s) = P(s) \ldots P(2\delta) P(1\delta) P(0) K(0) = \lim_{N\to\infty} \prod_{n=1}^{N} P(n\delta) K(0)$

This solution is nothing but parallel transport of $K(s) = P(s)$ along instantaneous eigenspaces $\lambda(s)$, a fold degenerate, i.e. parallel transport in $Gr(q, dim\, \mathcal{H})$

$\Rightarrow$ understanding: Parallel transport on sphere of tangential vectors $\in \mathbb{R}^2$

$\Rightarrow$ manipulation of initial vector

$\Rightarrow$ useful for adiabatic manipulation