Topology in condensed matter physics
Exercise sheet 9

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9.1 Spectrum and Diagonalization of the Su-Schrieffer-Heeger (SSH) Model

The Su-Schrieffer-Heeger (SSH) model is a fundamental (toy) model for a one-dimensional topological insulator. The SSH chain consists of N unit cells, here with periodic boundary conditions, each containing two sites A and B

$$\mathcal{H} = \sum_{j=1}^{N} \left(v\, c_{j,A}^{\dagger} c_{j,B} + w\, c_{j+1,A}^{\dagger} c_{j,B} + \text{h.c.} \right), \tag{1}$$

where $v > 0$ descirbes the intracell hopping, $w > 0$ the intercell hopping, and $c_{j,\alpha}^{\dagger}$ creates a fermion on sublattice $\alpha \in \{A, B\}$ of the j^{th} site.

a. (3 points) Use the Fourier transform of the operators

$$c_{j,\alpha} = \frac{1}{\sqrt{N}} \sum_{k} e^{\mathrm{i}kj} c_{k,\alpha}, \tag{2}$$

where $k = \frac{2\pi m}{N}$ with $m \in \{0, 1, \ldots, N-1\}$ to show that the Hamiltonian can be written as

$$\mathcal{H} = \sum_{k} \Psi_k^{\dagger} H(k) \Psi_k, \tag{3}$$

where $\Psi_k = \begin{pmatrix} c_{k,A} \\ c_{k,B} \end{pmatrix}$ is a two-component spinor, and identify the 2×2 momentum space Hamiltonian $H(k)$.

b. (3 points) Decompose the bulk Hamiltonian $H(k)$ in terms of the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{12}$$

such that $H(k) = d_0(k)I + \vec{d}(k) \cdot \vec{\sigma}$. Find the coefficients $d_0(k)$, $d_x(k)$, $d_y(k)$, and $d_z(k)$. Show that the energy dispersion relation is $E_{\pm}(k) = \pm|\vec{d}(k)|$ and determine under which parameters and at which momentum k the band gap closes.

9.2 Topology, Vector Visualization, and the Winding Number

Because $d_z(k) = 0$, the vector $\vec{d}(k) = (d_x(k), d_y(k), d_z(k))$ lies in the xy plane. As the momentum k varies continuously across the Brillouin zone $k \in [-\pi, \pi]$, the vector $\vec{d}(k)$ traces a closed curve in this plane.

a. (2 points) Describe the geometric shape of the curve traced by $\vec{d}(k) = (d_x(k), d_y(k))$ in the xy plane. Show how the trajectory behaves for the two phases: (i) $v > w > 0$ and (ii) $w > v > 0$. Geometrically argue whether the origin $(0, 0)$ is enclosed by this curve.

b. (4 points) The topological invariant characterizing this one-dimensional system is the winding number ν. It measures the number of times the vector $\vec{d}(k)$ wraps around the origin as k sweeps through the Brillouin zone:

$$\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d_x(k) \frac{d}{dk} d_y(k) - d_y(k) \frac{d}{dk} d_x(k)}{d_x(k)^2 + d_y(k)^2} dk \tag{24}$$

By defining a complex number $z(k) = d_x(k) + \mathrm{i} d_y(k)$, show that Eq. (24) can be written in the form

$$\nu = \frac{1}{2\pi \mathrm{i}} \int_{-\pi}^{\pi} \frac{1}{z(k)} \frac{dz(k)}{dk} dk \tag{25}$$

c. (4 points) Evaluate the winding number ν analytically for the two phases

(i) $v > w > 0$

(ii) $w > v > 0$

Hint: Write $z(k)$ in a form where you can pull out the dominant amplitude and use the expansion $\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$ for $|x| < 1$.

End