Topology in condensed matter physics
Exercise sheet 8

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8.1 Spectrum of Kitaev's Majorana chain Hamiltonian

Find the single-particle spectrum of the periodic Kitaev Majorana chain

$$\mathcal{H} = \sum_{j=1}^{N} \mu \left(c_j^\dagger c_j - 1/2 \right) + w c_j^\dagger c_{j+1} + w c_{j+1}^\dagger c_j + \Delta c_{j+1}^\dagger c_j^\dagger + \Delta^* c_j c_{j+1}, \tag{1}$$

with $c_{N+1} \equiv c_1$. One route is the following

a. (1 point) Use the spinor $\tilde{c}_j = (c_j, c_j^\dagger)$ to reexpress the Hamiltonian in matrix form.

b. (5 points) Conduct a Fourier transform by introducing the spinor $d_j = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{2\pi \mathrm{i}/N jk} \tilde{c}_j$ and conducting the sum over the real space index j. Hint: The sums are geometric series, and $\sum_{j=1}^{N} e^{\mathrm{i}ja} = N\delta_{0,a}$.

8.2 Reformulate a general fermionic Hamiltonian in Majorana form

Consider a general discrete noninteracting fermionic Hamiltonian

$$\mathcal{H} = \sum_{l,m=1}^{N} c_l^\dagger A_{l,m} c_m + c_l^\dagger B_{l,m} c_m^\dagger \ . \tag{9}$$

a. (3 points) Reformulate Eq. (9) in Majorana form

$$\mathcal{H} = \sum_{l,m=1}^{N} \gamma_l D_{l,m} \gamma_m. \tag{10}$$

Here, $\gamma_{2j} = \frac{1}{\sqrt{2}} \left(c_j + c_j^\dagger \right)$ and $\gamma_{2j+1} = \frac{1}{\sqrt{2}\mathrm{i}} \left(c_j - c_j^\dagger \right)$, for integers j. Show that the matrix D can be chosen real and skew-symmetric, i.e., $D^T = -D$.
Hint: *Use matrix notation*

8.3 Calculation with Pfaffians

In this exercise, we derive a formula for the Pfaffian of a $2n \times 2n$ skew-symmetric matrix A, the entries of which only depend on their offset from the diagonal:

$$A_{i,j} = A(j - i). \tag{11}$$

Here, $j - i$ has to be seen modulo $2n$. This formula is used in the lecture to derive the $\mathbb{Z}_2$ invariant of the Kitaev chain.

Hint: The following formulas may be useful

$$\begin{aligned}\mathrm{Pf}(A)^2 &= \det(A) && (12)\\ \mathrm{Pf}(BAB^T) &= \mathrm{Pf}(A)\det(B) && (13)\\ \mathrm{Pf}\left(A_1 \oplus A_2\right) &= \mathrm{Pf}(A_1)\,\mathrm{Pf}(A_2). && (14)\end{aligned}$$

Discrete Fourier transformation

The general definition for $U_{qi} is$

$$U_{qj} = \frac{1}{\sqrt{N}} e^{-i\frac{2\pi}{N}kj} \ ,$$

where we define $q := \frac{2\pi}{N}k$ and $k \in \{0, ..., N-1\}$. In the following computation, we have $N = 2n$ and therefore $q = \frac{\pi}{n}k$, with $k \in \{0, ..., 2n-1\}$.

a. (1 point) The Fourier transform of such a matrix is defined as $\tilde{A}(q) = \sum_j e^{\mathrm{i}qj} A(j)$ with $q \in BZ = \frac{\pi}{n}\{0, \dots, 2n-1\}$. On the other hand $U_{q,j} = \frac{1}{\sqrt{2n}} e^{\mathrm{i}qj}$ is the Fourier transform matrix. Show that $\left(UAU^T\right)_{q_1,q_2} = \delta_{q_1,-q_2}\tilde{A}(q_2)$.

b. (1 point) Derive

$$\mathrm{Pf}\begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix} = s\det(M) \tag{15}$$

for an arbitrary matrix $n \times n$ matrix M, where $s \in \{-1, 1\}$ is just a sign depending on n.

c. Now combine the previous results to derive

$$\mathrm{Pf}(A) \propto \prod_{0<q<\pi} \det\bigl(\tilde{A}(q)\bigr) \times \begin{cases} \prod_{q\in\{0,\pi\}} \mathrm{Pf}\left(\tilde{A}(q)\right) & n \text{ even} \\ \mathrm{Pf}\left(\tilde{A}(0)\right) & n \text{ odd.} \end{cases} \tag{16}$$

Hint: You need to reorder the submatrices of A in order to apply Eq. (14) and Eq. (15).

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