Topology in condensed matter physics
Exercise sheet 7

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7.1 Conventional time reversal operator and time reversal symmetry

This exercise shall provide some physical intuition on time reversal and time reversal symmetry. Imagine a system of spin-1/2 particles, the annihilation operators of which are given by $\Psi_\uparrow(x)$ and $\Psi_\downarrow(x)$, where x is a vector in real space. The "conventional" time reversal operator on this system is defined to be an antiunitary operator T with

$$\begin{aligned} T\Psi_\uparrow(x) &= \Psi_\downarrow(x)T, \\ T\Psi_\downarrow(x) &= -\Psi_\uparrow(x)T. \end{aligned} \tag{1}$$

Formulated with the two-component spinor $\Psi(x) = (\Psi_\uparrow(x), \Psi_\downarrow(x))$, this is written as $T\Psi(x) = \mathrm{i}\sigma_y\Psi(x)T$, where σ_y is the second Pauli matrix, and hence, in the single particle picture, T can be written as $T = \mathrm{i}\sigma_y\mathcal{K}$, where $\mathcal{K}$ denotes complex conjugation. To clarify further nomenclature: the time conjugation A' of an operator A is given by $A' = TAT^{-1}$.

Note: The Pauli matrices are $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, *and* $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *These matrices fulfill the algebra* $\sigma_i\sigma_j = \delta_{i,j}\underline{\underline{1}} + \mathrm{i}\sum_k \epsilon_{i,j,k}\sigma_k$, *where* $\underline{\underline{1}}$ *is the* 2×2 *identity matrix and* ϵ *the Levi-Civita symbol.*

1. (1 point) Show that the time evolution operator $U(t) = exp(-\mathrm{i}/\hbar Ht)$ obeys $TU(t)T^{-1} = U(-t)$ if the Hamiltonian H is time reversal symmetric. I.e., the time conjugation of $U(t)$ is $U(-t)$.

2. (1 point) In momentum space, the annihilation operators are $c_{k,\sigma} = \int_{\mathbb{R}} dx \frac{e^{\mathrm{i}kx}}{\sqrt{2\pi}}\Psi_\sigma(x)$. How does time reversal symmetry act on $c_{k,\sigma}$?

3. (3 points) What is the time reversal conjugation of the spin operators $J^\lambda(x) = \frac{1}{2}\Psi^\dagger(x)\sigma_\lambda\Psi(x)$ for the spin in x-, y-, and z-direction, respectively. Use matrix notation to simplify your calculations.

4. (1 point) Assume the single-particle Hamiltonian of spinful particles in a magnetic field $B(x) = \{B^x(x), B^y(x), B^z(x)\}$,

$$H = \int_{\mathbb{R}} dx\ \Psi^\dagger(x) v\left(-\mathrm{i}\partial_x\right)\Psi(x) + \sum_{\lambda\in\{x,y,z\}} B^\lambda(x)J^\lambda(x), \tag{2}$$

   In general, here, the dispersion relation v can be any polynomial in its argument. Putting together what was found above, show that H is (conventionally) time reversal symmetric if and only if $B = 0$ (vanishing magnetic field) and v is an even polynomial.

5. (1 point) Is the square of the conventional time symmetry operator positive or negative? Do we have Kramers' partners?

End