Topology in condensed matter physics
Exercise sheet 5

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5.1 Bogoliubov transformations

A Bogoliubov transformation is used to diagonalize a second quantized, quadratic Hamiltonian with anomalous terms. These are terms of the form creation operator times creation operator and annihilation operator times annihilation operator, which appear in the mean-field description of superconductors and in the solution of the interacting fermion problem in one dimension.
Given the annihilation operators c_i for i in a countable index set I, we introduce the transformed annihilation operator

$$d_j = \sum_{i \in I} A_{j,i} c_i + B_{j,i} c_i^\dagger, \tag{1}$$

for each $j \in I$.

1. Assume that the operators c_i for $i \in I$ are fermionic, i.e., $\{c_i, c_j\} = 0$, $\left\{c_i^\dagger, c_j\right\} = \delta_{i,j}$. Derive (necessary and sufficient) conditions for the matrices A and B such that the d_j are fermionic as well.
2. Assume that the operators c_i for $i \in I$ are bosonic, i.e., $[c_i, c_j] = 0$, $\left[c_i, c_j^\dagger\right] = \delta_{i,j}$. Derive (necessary and sufficient) conditions for the matrices A and B such that the d_j are bosonic as well.
3. Find a parametrization for the matrices A and B of the Bogoliubov transformation of a single bosonic mode (i.e., $I = \{1\}$).
4. (* points) Diagonalize

$$H = \alpha b^\dagger b + b^\dagger b^\dagger + bb, \tag{2}$$

where $\alpha \in \mathbb{R}$ and b is a bosonic annihilation operator, by finding a boson d such that

$$H = \epsilon d^\dagger d + \text{const.} \tag{3}$$

For which values of α is such a diagonalization impossible?
Hint: Use $[d, H] = \epsilon d = \epsilon(Ab + Bb^\dagger)$ to obtain linear equations for the coefficients A and B. Interpretation: If the Hamiltonian is not diagonalizable by a Bogoliubov transformation, it describes an unstable equilibrium.

5.2 Symmetry reduction of Hamiltonians

Consider the following set of Hamiltonians, a two-site tight-binding model for spinful fermions, for instance, describing the valence electrons of a binary molecule in a magnetic field

$$\mathcal{H} = BS_x + t \sum_{\sigma \in \{\uparrow,\downarrow\}} c_{1,\sigma}^\dagger c_{2,\sigma} + c_{2,\sigma}^\dagger c_{1,\sigma}, \tag{4}$$

with the real parameters $t, B \in \mathbb{R}$ describing electron transfer between the two sites and the magnetic field in x-direction, respectively, and $S_x = \frac{\hbar}{2} \sum_{i=1}^{2} c_{i,\uparrow}^\dagger c_{i,\downarrow} + c_{i,\downarrow}^\dagger c_{i,\uparrow}$ being the total x-spin of the particles.

1. Show that $\mathcal{H}$ conserves the total x-spin, i.e., $[\mathcal{H}, S_x] = 0$. Does that remain true for an inhomogeneous magnetic field in x-direction?
2. What is the corresponding unitary symmetry of the family of Hamiltonians?
3. Derive the single particle Hamiltonian by writing $\mathcal{H}$ in the form

$$\sum_{a,b=1}^{4} C_a^\dagger h_{a,b} C_b. \tag{5}$$

   with $c = (c_{1,\uparrow}, c_{2,\uparrow}, c_{1,\downarrow}, c_{2,\downarrow})$. Here, h is the single particle Hamiltonian.

4. Block diagonalizes the Hamiltonian by rotating the spin basis into x-direction. Can you represent this transformation and the result in the matrix formulation introduced in Eq. (5)? *Hint: This can be done by introducing the rotated quasi-particles* $\tilde{c}_{i,\uparrow} = \frac{1}{\sqrt{2}}(c_{i,\uparrow} + c_{i,\downarrow})$ *and* $\tilde{c}_{i,\downarrow} = \frac{1}{\sqrt{2}}(-c_{i,\uparrow} + c_{i,\downarrow})$. *(This has an insightful explanation. Which?)*

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