


# Topology in condensed matter physics

## Exercise sheet 4

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### 4.1 Homotopies and contractible curves

Consider the continuous maps from the circle to the plane  $f : \mathbb{S}_1 \rightarrow \mathbb{R}^2$ .

1. (2 points) Show that all these maps  $f$  are point contractible. This means they are homotopic to the constant map  $c : \mathbb{S}_1 \rightarrow \mathbb{R}^2$  with  $x \mapsto p$  for an arbitrary,  $x$ -independent  $p \in \mathbb{R}$ .
2. (1 point) Show that all these maps  $f$  are homotopic to each other. You have just shown that the fundamental group of the plane is trivial.

### 4.2 Fundamental group

In this exercise, we complete the proof that the fundamental group is a group. In the following, we consider  $X$  to be a topological space and  $x_0$  to be a fixed point in this space.

1. (1 point) Well-definedness. Show that the concatenation operation  $[f] + [g] = [f + g]$  with  $f, g : \mathbb{S}_1 \rightarrow X$  is “well-defined” for homotopy classes of loops. This is equivalent to saying that taking an  $f_1 \in [f]$  and a  $g_1 \in [g]$ , it follows that  $[f_1 + g_1] = [f + g]$ .  
*Reminder:  $f + g$  for  $f, g : [0, 1] \rightarrow X$  with  $f(0) = g(0) = f(1) = g(1) = x_0 \in X$  is defined as*

$$(f + g)(s) = \begin{cases} f(2s) & \text{for } s \leq 1/2, \\ g(2s - 1) & \text{for } s > 1/2. \end{cases} \quad (1)$$

2. (1 point) parametrization unimportant. Show that loops of different parametrization that preserve the direction are homotopic. A path  $f : I \rightarrow X$  is said to be of a different parametrization than  $g : I = [0, 1] \rightarrow X$ , if there is a homeomorphism  $p : I \rightarrow I$  such that  $f \circ p = g$ . The orientation remains preserved if  $p(0) = 0$  and  $p(1) = 1$ .
3. (1 point) neutral element. Show that every nonempty topological space has a point contractible loop.
4. (1 point) associativity. Show formally that  $[(f_1 + f_2) + f_3] = [f_1 + (f_2 + f_3)]$ .

### 4.3 Fermions and second quantization

Consider a system of spinful fermions (spin  $\uparrow$  and  $\downarrow$ ) that can occupy a single orbital. This means the Hilbert space of this quantum system is spanned by the four states  $|0\rangle$  (no fermion),  $|\uparrow\rangle = c_\uparrow^\dagger |0\rangle$  (single fermion with spin up),  $|\downarrow\rangle = c_\downarrow^\dagger |0\rangle$  (single fermion with spin down),  $c_\uparrow^\dagger c_\downarrow^\dagger |\uparrow\downarrow\rangle$  (one fermion with spin up, one with spin down), where  $c_\sigma^\dagger$  with  $\sigma \in \{\uparrow, \downarrow\}$  creates a fermion with spin  $\sigma$ . The fermionic anticommutation relations  $\{c_\sigma^\dagger, c_{\sigma'}\} = c_\sigma^\dagger c_{\sigma'} + c_{\sigma'} c_\sigma^\dagger = \delta_{\sigma, \sigma'}$  and  $\{c_\sigma, c_{\sigma'}\} = 0$  hold. The Hamiltonian is

$$\mathcal{H} = \mu (n_\uparrow + n_\downarrow) + B (n_\uparrow - n_\downarrow), \quad (2)$$

where  $n_\sigma = c_\sigma^\dagger c_\sigma$ ,  $\mu$  is the on-site energy, and  $B$  corresponds to a magnetic field.

1. (2 points) Show that  $|0\rangle, |\uparrow\rangle, |\downarrow\rangle$ , and  $|\uparrow\downarrow\rangle$  are pairwise orthogonal.
2. (2 points) Express the Hamiltonian in matrix form

$$\mathcal{H} = \sum_{j,k} c_j^\dagger h_{j,k} c_k, \quad (3)$$

i.e., find out the coefficients  $h_{j,k}$ .

3. (2 points) Find out the spectrum (the eigenvalues) of the matrix  $h$  and compare them to the spectrum of  $\mathcal{H}$ . Explain how to construct the spectrum of  $\mathcal{H}$  from the spectrum of  $h$ .
4. (2 points) Discuss if there are off-diagonal terms of the Hamiltonian  $\mathcal{H}$  that physically make sense.

**End**