

# Topology in condensed matter systems

## Exercise sheet 3

University of Hamburg   
Thore Posske

### 3.1 Separation axioms

Consider the following so-called separation axioms for topological spaces, which are properties that a topological space may or may not have.

- T0** For each pair of distinct points there is an open set that contains one of the points but not the other one (the points are topologically distinguishable).
- T2 (Hausdorff)** For each pair of distinct points there are two disjoint open sets, each containing exactly one of the points.
- T4 (normal Hausdorff)** For each pair of disjoint, closed sets there are two disjoint open neighborhoods of these sets, each containing exactly one of the closed sets.

**For further discussion: Separation axioms**

(T0) *Kolmogorov space*: For every pair of distinct points  $x, y \in X$ , with  $x \neq y$ , exists at least one neighbourhood  $U$ , which contains one of the points  $x \in U$  but not the other one  $y \notin U$ .

(T1) *Accessible/ Frechet* For every two points  $x, y \in X$ , with  $x \neq y$ , exists a neighbourhood  $x \in U$  and  $y \in V$  for each point, which does not contain the other point  $x \notin U, y \notin V$ . The neighbourhoods  $U$  and  $V$  do not need to be disjoint.

(T2) *Hausdorff*: For every two points  $x, y \in X$ , with  $x \neq y$ , exists a neighbourhood  $x \in U$  and  $y \in V$  for each point, which does not contain the other point  $x \notin U, y \notin V$ . The neighbourhoods are disjoint  $U \cap V = \emptyset$ .

(T3) *Regular Hausdorff*: For every point  $x \in X$  and every closed set  $A \in X$ , with  $x \notin A$ , exists a disjoint neighbourhood  $x \in U$  and  $A \subset V$ . The neighbourhoods are disjoint  $U \cap V = \emptyset$ , with  $U, V \in X$ .

(T4) *Normal Hausdorff*: For every two disjoint closed subsets  $A \cap B = \emptyset$ ,  $A, B \subset X$ , exist neighbourhoods  $A \subset U$  and  $B \subset V$ , which are disjoint  $U \cap V = \emptyset$ ,  $U, V \in X$ .

Furthermore, we can conclude:

- Every Hausdorff space also satisfies (T1)

$$(T_2) \Rightarrow (T_1)$$

- We can not directly get  $T_2$  or  $T_3$  from  $T_4$

$$(T_4) \not\Rightarrow (T_3) \not\Rightarrow (T_2) .$$

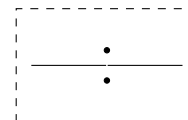
- A Hausdorff space that satisfies  $(T_4)$  also satisfies  $(T_3)$

$$(T_4) \wedge (T_2) \Rightarrow (T_3) .$$

1. Draw a schematics for the separation axioms  $T_0$ ,  $T_2$  and  $T_3$ .
2. Which separation axioms are valid for the following topological spaces?

(a)  $\mathbb{R}^2$

(b)  $X/\sim$ , where  $X = \mathbb{R} \times a \cup \mathbb{R} \times b$  and  $\sim: (x, a) \sim (x, b)$ , if  $x \neq 0$ . *schematics:*



(c) The graph of  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $x \mapsto \sin(1/x)$  as a subspace of  $\mathbb{R}^2$ .

(d)  $(X, \mathcal{T})$  with  $X = \{a, b, c\}$ ,  $\mathcal{T} = \{\{a, b, c\}, \emptyset, \{a\}, \{a, b\}\}$ .

(e)  $(X, \mathcal{T})$  with  $X = \{a, b, c\}$ ,  $\mathcal{T} = \{\{a, b, c\}, \emptyset\}$ .

## Definition of a manifold

For your information: a Hausdorff space (i.e., a topological space fulfilling  $T_2$ ), which is locally homeomorphic to  $\mathbb{R}^n$  for some positive integer  $n$  and has a countable base for its topology is called **manifold**. Manifolds are of central importance to the mathematical description of physics. Almost every space you have encountered during your studies can be seen as a manifold, including finite dimensional vector and Hilbert spaces, the phase space, and, of course, spacetime.

## 3.2 Representations of topological spaces

Draw (embeddings of) the following topological spaces. Every time there is no topology specified, assume the subspace topology of  $\mathbb{R}^n$ .

### Homeomorphism

A *homeomorphism* is a bijective map  $f : (X_1, \mathcal{T}_1) \rightarrow (X_2, \mathcal{T}_2)$  between topological spaces, where the map  $f$  and the inverse map  $f^{-1}$  is continuous.

If there is a homeomorphism, the topological spaces are called *homeomorph* to each other.

In the following exercise, we depict spaces that are homeomorphic to each other.

1. (1 point)  $\mathbb{R} \setminus \{0\}$ ;
2. (1 point)  $\mathbb{S}^1 \times \mathbb{S}^1$ , where  $\mathbb{S}^1$  is the circle;

### Torus (pl.: Tori)

In general, the torus is defined as

$$\mathbb{T}^n = \overbrace{\mathbb{S}^1 \times \dots \times \mathbb{S}^1}^n .$$

3. (1 point)  $X / \sim$ , where  $X = [0, 1] \times [0, 1]$  and  $x \sim y$  if (and only if) 1)  $x = y$ , 2)  $x = (x_1, 1)$  and  $y = (x_1, 0)$  for all  $x_1 \in [0, 1]$  or 3)  $x = (1, y_1)$  and  $y = (0, y_1)$  for some  $y_1 \in [0, 1]$ ;
4. (2 points)  $X / \sim$ , where  $X = \mathbb{R}^2 \setminus \{(0, 0)\}$  and  $x \sim y$  if (and only if)  $x = y$  or if the line joining  $x$  and  $y$  would go through the origin  $\{(0, 0)\}$ . This space is an example of a projective space.

**End**