Topology in condensed matter
Exercise sheet 2

University of Hamburg
Thore Posske

2.1 Metric spaces

Let (X, d) be a metric space.

1. (1 point) Show that the positivity $d(x, y) \geq 0$ of a metric follows from the remaining three axioms.
2. (2 points) Show that the identity of indiscernible follows from the axioms of a metric space:

$$d(x, y) = 0 \Leftrightarrow x = y \ .$$

3. (2 points) An open set $U \subset X$ is a set with all elements $x \in U$ having an open ball that contains x inside U. Show that an open set in a metric space is the union of open balls.
4. (2 points) Show that the "induced topology" of a metric space (as defined in the lecture) is a topology. *Reminder:* The natural topology of a metric space includes all sets that are unions of open balls and the empty set (or balls of radius $r = 0$ allowed).
5. (2 points) Construct a set that is not an open set but can be represented as the intersection of open sets. *Hint: Think, e.g., about infinite sections of open balls in a metric space.*

2.2 Connection between equivalence relation and partition

An equivalence relation $\sim$ on a set X can be defined as a map $\sim: X^2 \to \{\text{true}, \text{false}\}$ with the following properties[1]. Let $x, y, z \in X$, then:

reflexivity $x \sim x$.

symmetry If $x \sim y$, then $y \sim x$.

transitivity If $x \sim y$ and $y \sim z$, then $x \sim z$.

An equivalence class $[x]$ for $x \in X$ is the subset of all elements of X, that are equivalent to x.

1. (2 points) Consider a partition $\mathcal{P}$ of X. That is a family of disjoint subsets of X, the union of which is X. A partition induces a relation between two elements of X by $\sim: X^2 \to \{\text{true}, \text{false}\}$ by $x \sim y$ exactly then, if $\mathcal{P}$ contains a set, which possesses x as well as y. Show that this relation is an equivalence relation
2. (1 point) Show that the family of equivalence classes of an equivalence relation constitutes a partition.

You have just shown that an equivalence relation can be interpreted as a partition and vice versa.

End

[1] Usually one writes $x \sim y$ *(read "x is equivalent to y")* instead of "$\sim (x, y)$ *is true* ".