Topology in condensed matter physics
Exercise sheet 12

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12.1 Two indiscernible particles on a ring

Derive the configuration space for two indiscernible particles on a ring $\mathcal{C}_2^{\mathbb{S}_1} = ((\mathbb{S}_1 \times \mathbb{S}_1 \setminus D)/S_2$ by, for instance, repeating the steps from the lecture that led to the configuration space of two particles in $1D$, $2D$, and $3D$. As a reminder: here $D = \{(a,a)|a \in \mathbb{S}_1\}$ and S_2 is the symmetric group with two elements (i.e., permutations of two elements).

12.2 Linear representations up to a phase

When developing the quantum theory for non-simply connected spaces, a one-dimensional, complex "representation up to a phase" of the fundamental group of the space in question is needed. Consider X to be a path-connected topological space. Such a representation $\gamma : \pi_1(X) \to \mathbb{C}$ needs to fulfill

$$\gamma(f+g) = \gamma(f)\gamma(g)e^{\mathrm{i}\phi}, \tag{1}$$

$$|\gamma(g)| = 1. \tag{2}$$

with some constants $\phi \in [0, 2\pi)$ for all $f, g \in \pi_1(X)$.

a. Show that $\gamma : \mathbb{Z} \to \mathbb{C}$ with $\gamma(z) = e^{\mathrm{i}(\varphi z - \phi)}$ fulfills the above requirements. These are possible representations used in the quantum mechanical description of 2 anyons in 2D or for one particle on a ring. The parameter $\varphi \in [0, 2\pi)$ is called the statistical angle and describes the phase a particle picks up when traveling a path corresponding to the group element 1.

b. Show that for $\pi_1(X) = \mathbb{Z}_2$, i.e., the group with exactly two elements, there are no other such representations than $\gamma_1, \gamma_2 : \mathbb{Z}_2 \to \mathbb{C}$ with $\gamma_1(z) = 1$ and $\gamma_2(z) = (-1)^z$ (up to an additionally multiplied phase factor). This is to say: in three dimensions, there are only bosons and fermions, but no anyons.

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