Topology in condensed matter physics
Exercise sheet 11

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11.1 Quantum physics of indiscernible particles

a. (1 point) Find at least one mistake in the following textbook like the derivation of the existence of only bosons (symmetric wave function) and fermions (anti-symmetric wave function), which is still frequently printed in textbooks and articles, [see, e.g., Nature **637**, 314 (2025) Eqs. (1) and (2)]:
Consider the wave function of an N particle system of indiscernible particles (for simplicity without spins and additional degrees of freedom) in spatial representation $\Psi : \mathbb{R}^{N\times d} \to \mathbb{C}$, where d is the dimension of space. A permutation of two particles, x_i and x_j $(i \neq j)$, is represented by the exchange of the respective coordinates in the wave function. Then, in general,

$$\Psi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = c\Psi(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n). \tag{1}$$

We now exchange x_i and x_j on the right-hand side again and obtain

$$\Psi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = c^2\Psi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n). \tag{2}$$

We conclude that $c = 1$ and $c = -1$ are the only possibilities for c.

11.2 Fun with braids

The braid group of order $n+1$ is generated by σ_i, where $i \in \{1 \ldots n\}$, which obey the following identities

$$\left(\sigma_i \sigma_j \sigma_i^{-1} \sigma_j^{-1}\right) = 1 \quad \text{if } |i-j| > 1, \tag{3}$$

$$\left(\sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1}\right) = 1. \tag{4}$$

a. (2 points) Draw these relations with a "braid diagram".

b. (2 points) Set $\Delta = \sigma_1 \sigma_2 \sigma_1$. Show that

$$\sigma_1 \Delta = \Delta \sigma_2, \tag{5}$$

$$\sigma_2 \Delta = \Delta \sigma_1, \tag{6}$$

and hence that Δ^2 commutes with both σ_1 and σ_2.

11.3 Configuration space — alternative view

The configuration space of indiscernible particles has been defined in the lecture by taking the set of tuples in $\mathbb{R}^d$, where d is the spatial dimension, excluding tuples that include equal points and identifying tuples that result as permutations from each other. Here, we want to show that a different philosophy leads to the same result.

If n discernible particles are described, they can be labeled and hence their positions be collected into an n-tuple. If indiscernible particles shall be described, the labels are obsolete but still the n positions can be collected into a set. Sets naturally forget about the ordering of their elements, and naturally treat equal elements as one. Hence, consider the family $\mathcal{F}$ of sets with n elements, each element belonging to $\mathbb{R}^d$, where $d \in \{1, 2, 3\}$ is the dimension of space. For notational purposes, we also introduce the tuples with n pairwisely differently elements $\mathcal{G} = \{x \in \left(\mathbb{R}^d\right)^n \mid \text{ with } x_j \neq x_i \text{ if } j \neq i\}$.

Now regard the map $q : \mathcal{G} \to \mathcal{F}$, which forgets about the order, i.e., $q\left(x_1, x_2, \ldots, x_n\right) = \{x_1, x_2, \ldots, x_n\}$. Take $\delta_x = \min_{i,j \in \{1,\ldots,n\}, i \neq j} \frac{1}{\sqrt{2}} \left(|x_i - x_j|\right)$ to be the minimal distance between two positions.

a. (1* points) Show that $q^{-1}(q(B_\delta(x)))$, where $x = (x_1, \ldots, x_n)$ is a tuple with pairwise different elements, is a direct sum of homeomorphic sets.

We furthermore take $\mathcal{U} = \{q(B_{\gamma_x}(x)) \mid x \in \mathcal{G}, \gamma_x \in \mathbb{R} < \delta_x\}$ as a base to define the topology of $\mathcal{F}$. I.e., the topology $\mathcal{T}$ of $\mathcal{F}$ is constructed by all possible unions of elements of $\mathcal{U}$.

b. (2* points) Show that the so defined topology of $\mathcal{F}$ is the finest topology such that q is continuous. Subsequently show that therefore there is an isomorphism between the notion of a configuration space as defined in the lecture and as defined here.

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