Topology in condensed matter physics

Exercise sheet 10

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10.1 Adiabatic time evolution

Adiabatic time evolution is described by a linear, first-order differential equation for operators on a Hilbert space $\mathcal{H}$, which reads

$$K'(s) = [P'(s), P(s)]\, K(s), \tag{1}$$

where $P(s)$ is a smoothly varying projector onto a subspace of $\mathcal{H}$. Usually, this subspace is thought of as the instantaneous ground space of some Hamiltonian H. The solution Eq. (1) is chosen to fulfill the initial condition $K(0) = P(0)$.

a. Show that $K(s) = P(s)\Omega(s)$, where $\Omega(s) = \mathcal{T}\left\{e^{\int_0^s ds' A(s')}\right\}$, is a solution of Eq. (1). From the lecture, we know that $K(s) = \Omega(s)P(0)$ is also a solution of Eq. (1). Since the solution is uniquely determined, we have just shown that $P(s)\Omega(s) = \Omega(s)P(0)$, which results in $P(s)K(s) = K(s) = K(s)P(0)$ and the time evolution of the projection operator $P(s) = \Omega(s)P(0)\Omega^{-1}(s)$.

10.2 Topological quantum computing with Majoranas

In this exercise, you show how the degenerate ground state in systems with Majorana modes can be adiabatically manipulated. Consider four Majorana operators γ_j with $j \in \{1 \dots 4\}$ in a system with tunable on-site potentials

$$\mathcal{H}(t) = \mathrm{i}\left(\mu_b(t)\gamma_3\gamma_4 + \mu_r(t)\gamma_2\gamma_3 + \mu_l(t)\gamma_1\gamma_3\right). \tag{2}$$

a. (2 points) The adiabatic evolution of the Hamiltonian starts at $t = 0$ with $\mu_b(0) = \mu > 0$ and $\mu_l(0) = \mu_r(0) = 0$. What is the ground space's energy and what is its degeneracy?

b. (2 points) Calculate the projector to the instantaneous ground space for general values of s with $\mu_b(s) = \cos(s)\mu$, $\mu_r(0) = \sin(s)\mu$ and $\mu_l(0) = 0$. Hint: *One brute force possibility is to use the many-body basis* $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$*, diagonalize the Hamiltonian, and find the projectors. Another one is to diagonalize the Hamiltonian in Majorana basis, and work out the projector in second quantization.*

c. ($4^\dagger$ points) [1] Solve Kato's adiabatic evolution for $\mu_b(s) = \cos(s)\mu$, $\mu_r(0) = \sin(s)\mu$ and $\mu_l(0) = 0$ for s

[1] † The dagger marks extra hard exercises.

from 0 to $\pi/2$.

$$K'(s) = P'(s)K(s). \tag{3}$$

d. (6 points) Combine the adiabatic operators of the following three-step evolution to get the "braiding operator", i.e., $K(3\pi/2)$, for

1. $\mu_b(s) = \cos(s)\mu$, $\mu_r(0) = \sin(s)\mu$ and $\mu_l(0) = 0$, for $s \in [0, \pi/2]$.
2. $\mu_b(s) = 0$, $\mu_r(0) = \sin(s)\mu$ and $\mu_l(0) = -\cos(s)\mu$ for $s \in [\pi/2, \pi]$.
3. $\mu_b(s) = -\sin(s)\mu$, $\mu_r(0) = 0$ and $\mu_l(0) = -\cos(s)\mu$ for $s \in [\pi, 3/2\pi]$.

Can you visualize the process? What is its physical meaning? Think, e.g., about the Majoranas being at the end of Kitaev Majorana chains (whose bulk is not considered here because it is gapped from the low-energy theory).

e. (3* points) [2] What does the braiding operator correspond to when expressed in the many-body basis $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$? Is it an Abelian or non-Abelian operator? Extra question: What if two additional Majorana modes are added that do not participate in the Hamiltonian?

Remark on the nonabelian braiding of two free Majoranas

Note that the actual many body space $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ separates into "superselection sectors" with different parity, because the Hamiltonian & no observables mixes fermionic parities

$$H_A = \text{span}\{|00\rangle, |11\rangle\}, \quad H_B = \text{span}\{|01\rangle, |10\rangle\} \ .$$

Therefore, the ground space merely obtains a phase by the adiabatic evolution. This changes if we introduce a larger degeneracy with additional Majorana modes γ_5 & γ_6 that correspond to a zero mode (the Hamiltonian and hence the Kato propagator remains unchanged).

Similarly, we can say that

$$[B_{12}, B_{21}] = [B_{12}, -B_{12}] = 0$$

but $[B_{12}, B_{25}] \neq 0$, i.e., the group of braiding operators (proof that they form a group) is only non-Abelian if sufficiently many braiding operators exist.
Hence, we need at least four free Majoranas for adiabatic state manipulation and topological quantum computing.

End

[2] *: The asterisk marks volunatary exercises.