Topology in Condensed Matter — Cheat Sheet

based on the lectures of T. Posske, first version by A. Junker.

1. Mathematical Foundations

Open ball	$B_r(x) = \{y \in X \mid d(x,y) < r\}$
Interior	$\overset{\circ}{A} = \bigcup\{U \in \mathcal{T} \mid U \subseteq A\}$
Homotopy class	$[f] = \{g : X \to Y \mid g \simeq f\}$
Equivalence class	$[x]_\sim = \{y \in X \mid x \sim y\}$
Projective space	$\mathbb{RP}^n = \mathbb{S}^n / \sim$
Metric topology	$\mathcal{T} = \{U \subset X \mid \forall x \in U\ \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset U\}$
Top. subspace	$\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}_X\}$
Direct sum top.	$\mathcal{T} = \{\sqcup_{i\in I} U_i \mid U_i \in \mathcal{T}_i\}$
Quotient top.	$\mathcal{T}_{X/\sim} = \{U \subset X/\sim \mid q^{-1}(U) \in \mathcal{T}_X\}$ $q : X \to X/\sim,\ x \mapsto [x]$
Quotient space	$X/\sim = \{[x] \mid x \in X\}$
Fund. group	$\pi_1(X, x_0) = \{[f] \mid f : [0,1] \to X,$ $f(0) = f(1) = x_0\}$

2. The 10-Fold Way Table

Symmetry				$\pi_d(\mathcal{G}(n+m,m))$			
T^2	P^2	C^2	Class	$d=1$	2	3	4
0	0	0	**A**	–	$\mathbb{Z}$	–	$\mathbb{Z}$
0	0	1	**AIII**	$\mathbb{Z}$	–	$\mathbb{Z}$	–
1	0	0	**AI**	–	–	–	$\mathbb{Z}$
1	1	1	**BDI**	$\mathbb{Z}$	–	–	–
0	1	0	**D**	$\mathbb{Z}_2$	$\mathbb{Z}$	–	–
-1	1	1	**DIII**	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	–
-1	0	0	**AII**	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$	–
-1	-1	1	**CII**	$\mathbb{Z}$	–	$\mathbb{Z}_2$	$\mathbb{Z}_2$
0	-1	0	**C**	–	$\mathbb{Z}$	–	$\mathbb{Z}_2$
1	-1	1	**CI**	–	–	$\mathbb{Z}$	–

3. Fundamental Symmetries

Example representations / specific operators

Time-Reversal (T)	$T\,\alpha\, c_\uparrow^\dagger T^{-1} = \alpha^* c_\downarrow^\dagger,$ $T\,\alpha\, c_\downarrow^\dagger T^{-1} = -\alpha^* c_\uparrow^\dagger$ $T = i\sigma_y \mathcal{K}$ (spin-1/2)
Particle-Hole (P)	$P \alpha c_\sigma P^{-1} = \alpha^* c_\sigma^\dagger$ (BdG)
PTC Relations	$C = T \cdot P, \quad T = C \cdot P,$ $P = C \cdot T$

4. Quantum Operators & Transformations

Bosons	$[b_i, b_j^\dagger] = \delta_{ij}, \quad [b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0$
Fermions	$\{c_i, c_j^\dagger\} = \delta_{ij}, \quad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0$
Majorana	$\gamma_j^A = \frac{1}{\sqrt{2}}(c_j + c_j^\dagger), \quad \gamma_j^B = \frac{1}{i\sqrt{2}}(c_j - c_j^\dagger)$ $\{\gamma_i, \gamma_j\} = \delta_{ij}, \quad \gamma_i^2 = 1, \quad \gamma_i\gamma_j = -\gamma_j\gamma_i\ (i \neq j)$
Backtrans.	$c_j = \frac{1}{\sqrt{2}}(\gamma_j^A + i\gamma_j^B), \quad c_j^\dagger = \frac{1}{\sqrt{2}}(\gamma_j^A - i\gamma_j^B)$
FT (Forward)	$c_j = \frac{1}{\sqrt{N}} \sum_k e^{ikj} c_k$
FT (Inverse)	$c_k = \frac{1}{\sqrt{N}} \sum_j e^{-ikj} c_j, \quad k = \frac{2\pi m}{N}$

5. General Equations

Winding No. $\quad Z_g = \frac{1}{L}\int_{t_a}^{t_e} g'(t)\,dt$

Pfaffian $\quad \mathrm{Pf}(A) = \frac{1}{2^n n!}\sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma)\prod_{i=1}^{n} A_{\sigma(2i-1),\sigma(2i)}$

Gen. Ham.

$$H = \sum_{l,m=1}^{N}\left(c_l^\dagger \tilde{A}_{lm} c_m + c_l^\dagger \tilde{B}_{lm} c_m^\dagger + \text{h.c.}\right)$$

$$H = \frac{i}{4}\sum_{l,m=1}^{2N} \gamma_l' D_{lm} \gamma_m' + \text{const.}$$

$$\gamma_l' = \begin{cases} \gamma_{l/2}^B & \text{for } l \text{ even} \\ \gamma_{(l+1)/2}^A & \text{for } l \text{ odd} \end{cases}$$

Kitaev Chain

$$H = \mu \sum_{j=1}^{N}\left(c_j^\dagger c_j - \frac{1}{2}\right) + \sum_{j=1}^{N-1}\left(w c_j^\dagger c_{j+1} + \Delta c_j^\dagger c_{j+1}^\dagger + \text{h.c.}\right)$$

6. Adiabatic Theorem & Path Integrals

Cond. adia.	$K(s) = U_\tau(s)P(0) + \frac{1}{i\tau}U_\tau(s)\int_0^s U_\tau'(s')S(s')K(s')K'(s)\,ds$
Kato Propagator	$K(s) = \mathcal{T}e^{\int_0^s A(s')K(s')ds'}P(0)$
Properties	$K(s)P(0) = K(s), \quad P(s)K(s) = K(s)$
Kato Eq. (K.E.)	$\frac{d}{ds}K(s) = A(s)K(s) = \Big(\frac{d}{ds}P(s)\Big)K(s)$
Sol. to K.E.	$K(s) = \lim_{\substack{N\to\infty \\ N\delta=s}} \prod_{n=1}^{N} P(n\delta)K(0)$
Time Evol.	$U_\tau(t) = \mathcal{T}\exp\left\{-\frac{i}{\hbar}\int_0^t H_\tau(t')\,dt'\right\}$
Time Ordering	$\mathcal{T}\big(A(t_1)B(t_2)\big) = \begin{cases} A(t_1)B(t_2) & t_1 > t_2 \\ \pm B(t_2)A(t_1) & t_2 < t_1 \end{cases}$
Total Evol.	$\lvert\psi(t)\rangle = e^{-i\theta}e^{i\gamma[C]}\lvert\psi(0)\rangle$
Braiding Ham.	$H = \mu_b(s)\gamma_3\gamma_4 + \mu_r(s)\gamma_2\gamma_3 + \mu_l(s)\gamma_1\gamma_2$
Proj. Rep.	$\gamma : \pi_1(X) \to \mathbb{C},\ \gamma(g+g') = \gamma(g)\gamma(g')e^{i\phi}$
Path Integral	$K(x_a, t_a; x_e, t_e) = \sum_{g\in\pi_1(X)} \gamma(g)\int \mathcal{D}[q]e^{\frac{i}{\hbar}S[q]}$

7. Configuration Spaces

$$\mathcal{C}_2^1 = (\mathbb{R}^2_{\geq 0} \setminus D)/\sim$$

$$\mathcal{C}_2^2 = (\mathbb{R}^4 \setminus D)/\sim, \quad (r, R) \in \mathbb{R}^2 \times \mathbb{R}^2$$

Center of Mass $R = \frac{x_1 + x_2}{2}, \quad r = \frac{x_1 - x_2}{2}$

$$\mathcal{C}_2^3 = \Big(\mathbb{R}^3 \times ([0,\infty) \times \mathbb{S}^2) \setminus D\Big)/\sim$$

8. Useful Relations & Pauli Matrices

$$\begin{aligned}
&[AB, C] = A[B, C] + [A, C]B \\
&[A, BC] = [A, B]C + B[A, C] \\
&\{AB, C\} = A[B, C] + \{A, C\}B \\
&\{A, BC\} = [A, B]C + B\{A, C\} \\
&[AB, C] = A\{B, C\} - \{A, C\}B \\
&[AB, CD]_\pm = A[B, C]_- D + AC[B, D]_- \\
&\qquad\qquad + [A, C]_- DB + C[A, D]_\pm B \\
&\sigma_i\sigma_j = \delta_{ij}I + i\sum_k \epsilon_{ijk}\sigma_k \\
&(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) = (\mathbf{a}\cdot\mathbf{b})I + i(\mathbf{a}\times\mathbf{b})\cdot\boldsymbol{\sigma}
\end{aligned}$$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$