

RIKEN tutorial 2026

Topological matter & Majoranas

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www.posske.de → Lectures → RIKEN 2026

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Content

1) Motivation

- Content of workshop & tutorial
- Definition topological phase & bulk boundary correspondence

2) Topological classification of noninteraction electronic matter

- Wigner's theorem: The fundamental symmetries in quantum mechanics & antiunitary symmetries
- Time reversal, particle hole, & chiral symm.
- The 10-fold way

3) Kitaev-Majorana chain

- Special cases, Majoranas
- Topological invariant

Resources
Collected for you at

www.posske.de → Lectures → RIKEN 2026

<https://www.posske.de>

Kitaev 2001

Kitaev 2003

Ryu, Schnyder, Furusaki, Ludwig '09, '10.

⋮

Review helical liquids

Hsu, Stan, Klinovaja, Loss 2021

1. Motivation

- Preparation for workshop
see workshop content
→ Majoranas, Majorana variants (poor man's quantum spin liquids (where Majoranas also play a central role))
- Expand your research scope
- Understand physics relevant likewise for fundamental aspects (own antiparticle, nonabelian anyons, etc.) as well as for applied aspects (sensors, industry-relevance quantum information technologies, etc., see also contributions of Dmitry Pikhulin & Roman Lutchyn)

1.1. Fundamentals

Noninteracting electronic matter

$$\mathcal{H} = \sum_{j,l} A_{jl} c_j^\dagger c_l + B_{jl} c_j^\dagger c_l^\dagger + B_{jl}^* c_l c_j$$

$\hookrightarrow$ quantum numbers,
also continuum possible (sum $\rightarrow$ integral)

$c_j^\dagger$ is fermionic creation operator
 $\{c_j^\dagger, c_l\} = \delta_{jl}$, $\{c_j^\dagger, c_l^\dagger\} = 0$.

Why "noninteracting"?

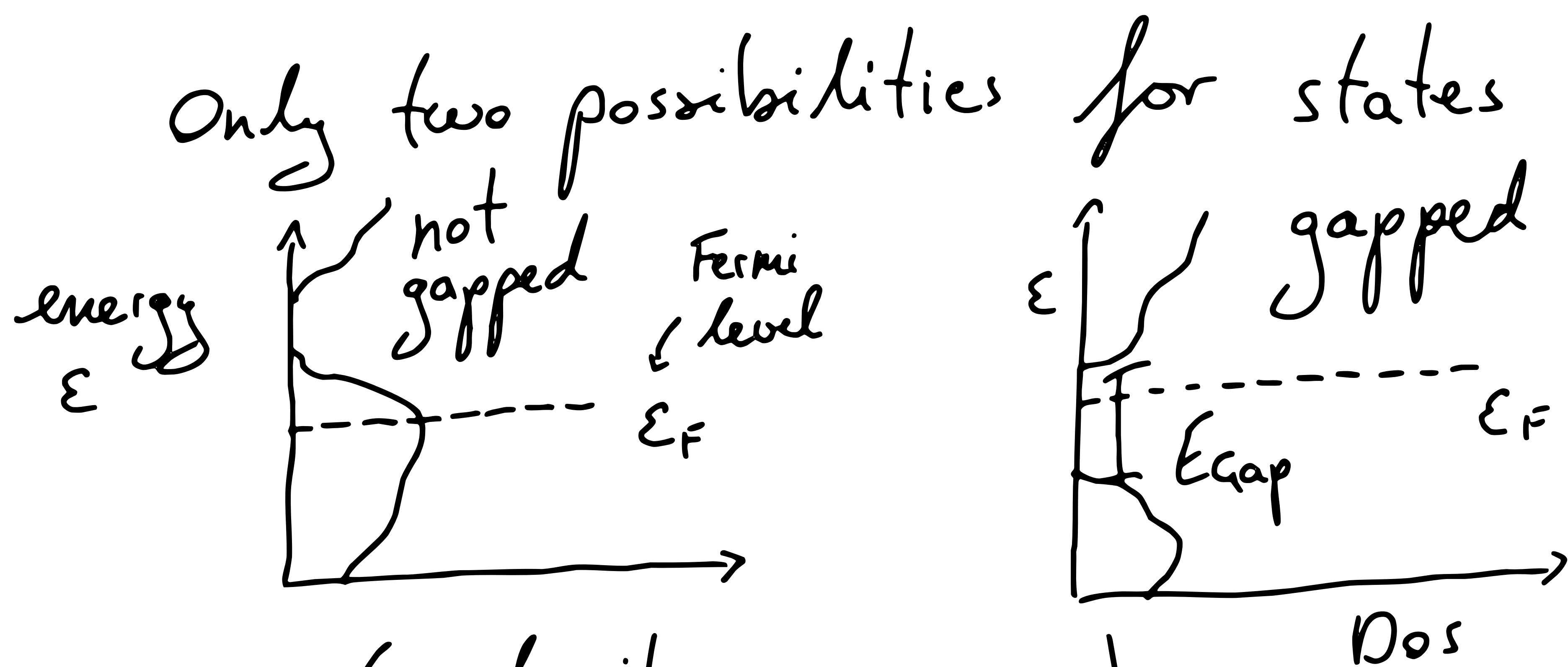
[see Volovik ('03)
for interacting
electrons]

- Single-particle Hamiltonian
- No interaction terms like $c_l^\dagger c_l c_j^\dagger c_j$.
- Introduce normal mode
 $d_j = \sum_l C_{jl} c_l + D_{jl} c_l^\dagger$ (with tailored
matrices D & C)

$$\Rightarrow \mathcal{H} = \sum_j \varepsilon_j d_j^\dagger d_j \text{ is diagonal}$$

$\Rightarrow$ Noninteraction fermionic normal modes

$\hookrightarrow$ see Bogoliubov transformation for more
background. Exercise on webpage.



density of states (Dos)

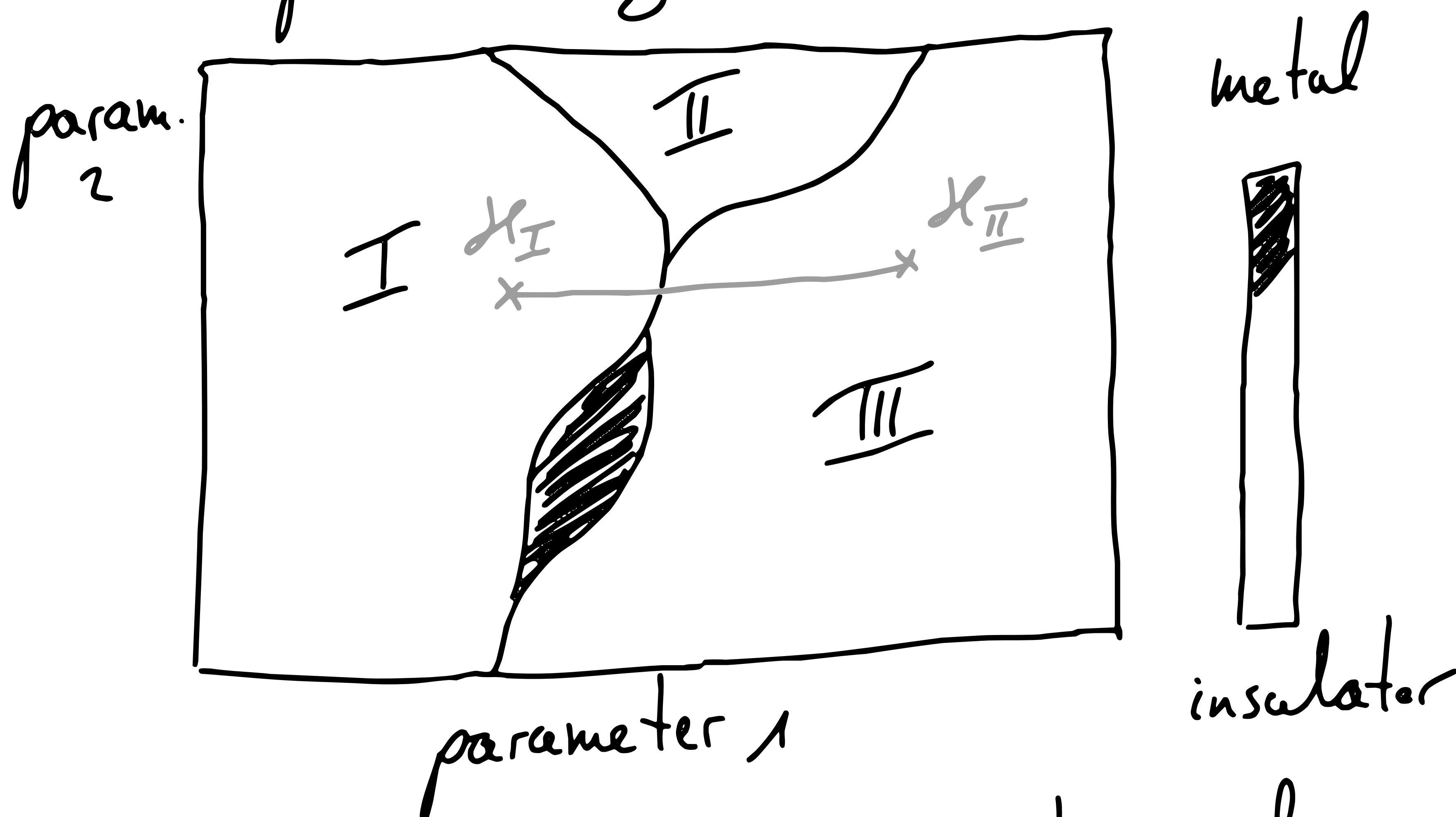
Metal

Insulator & superconductor

1.2. Topological phases

$\Rightarrow$ two electronic phases only?

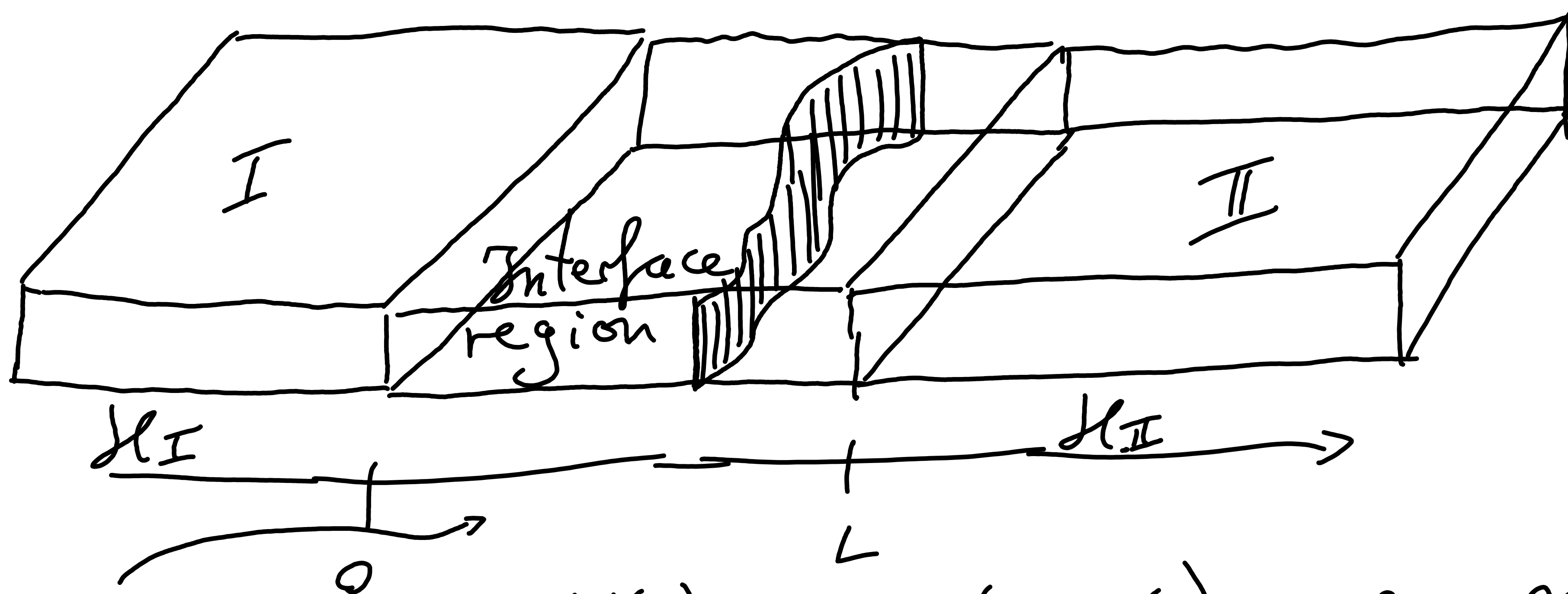
phase diagram



Finding: Topological distinction of gapped phases. Definition: Two systems in same phase if (& only if) parameters can be continuously deformed without closing electronic $\leftarrow$ gap.

1.3. Bulk boundary correspondence

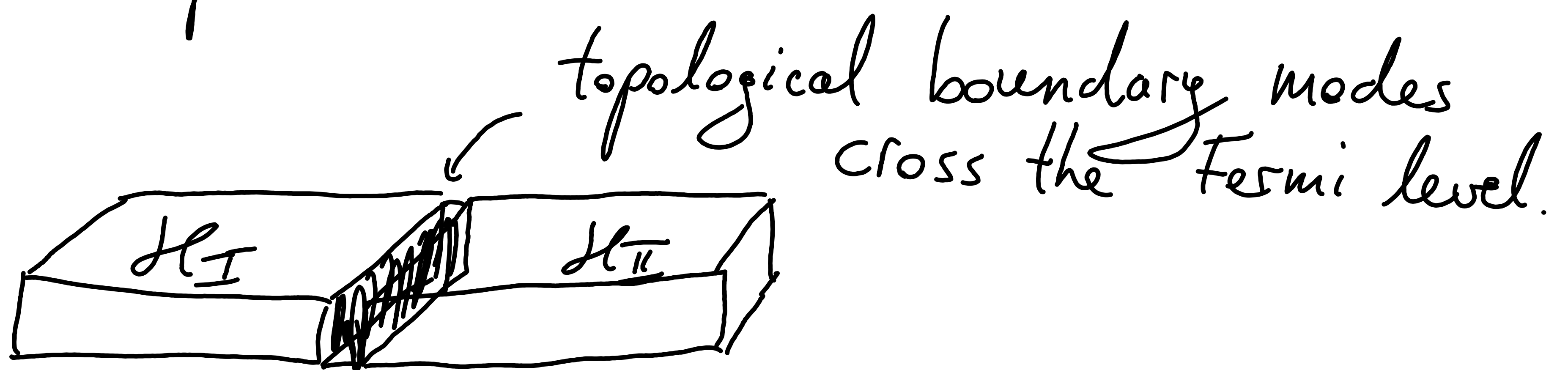
Physical consequence of connected different topological phases



$$\mathcal{H}(x) = \mathcal{H}_{II}\left(\frac{x}{L}\right) + \mathcal{H}_{I}\left(1 - \frac{x}{L}\right), \quad \text{see path in previous diagram}$$

$\Rightarrow$ somewhere metallic region at interface.

Zoom out



Attention: Numerous (topologically trivial) interface modes exist. Hard to distinguish $\rightarrow$ see workshop talks

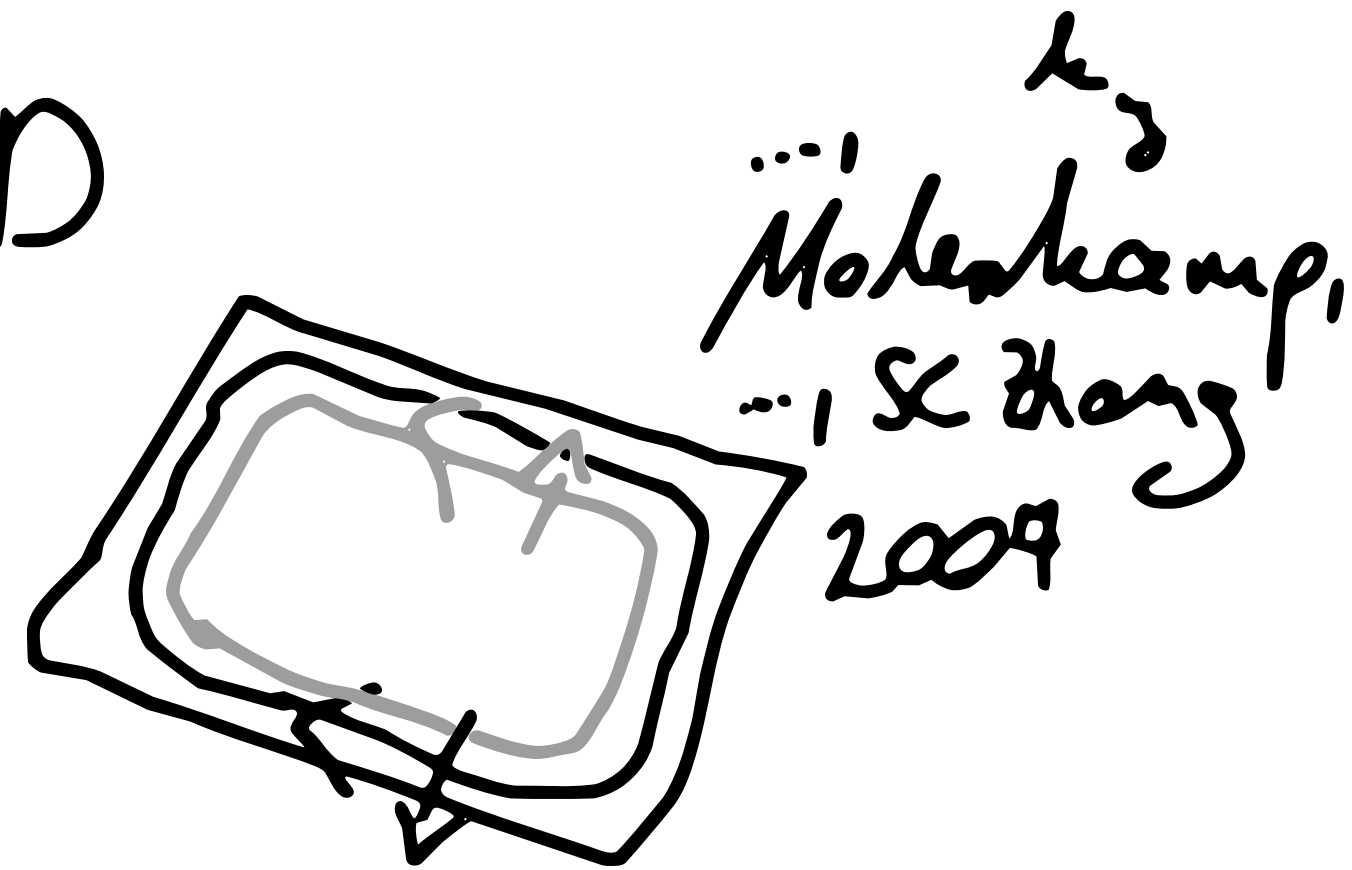
Hint: See additional online resources of Nielsen-Ninomiya theorem & bulk boundary correspondence & Atiyah-Singer index theorem ('63)

Examples for topological boundary modes

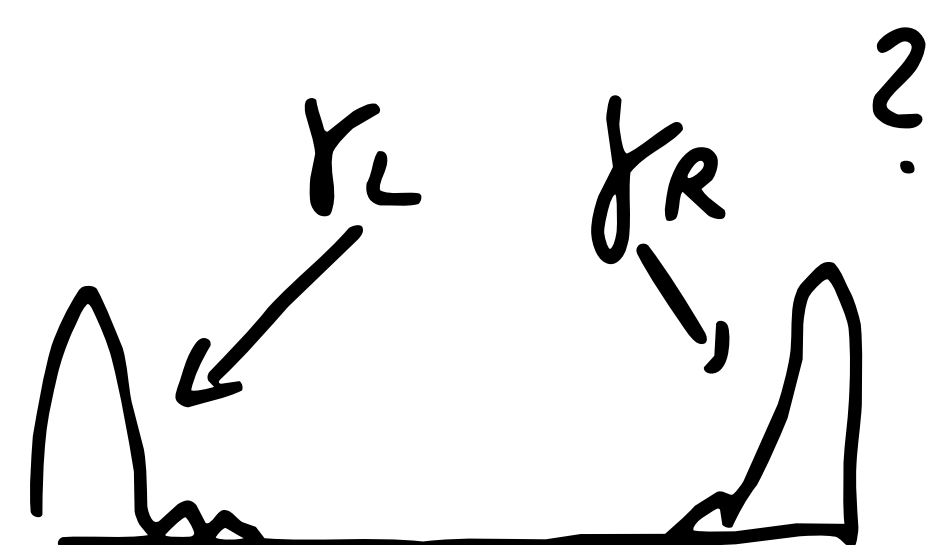
Quantum Hall effect, 2D
↳ edge channels (chiral)



Quantum spin Hall effect, 2D
(helical)



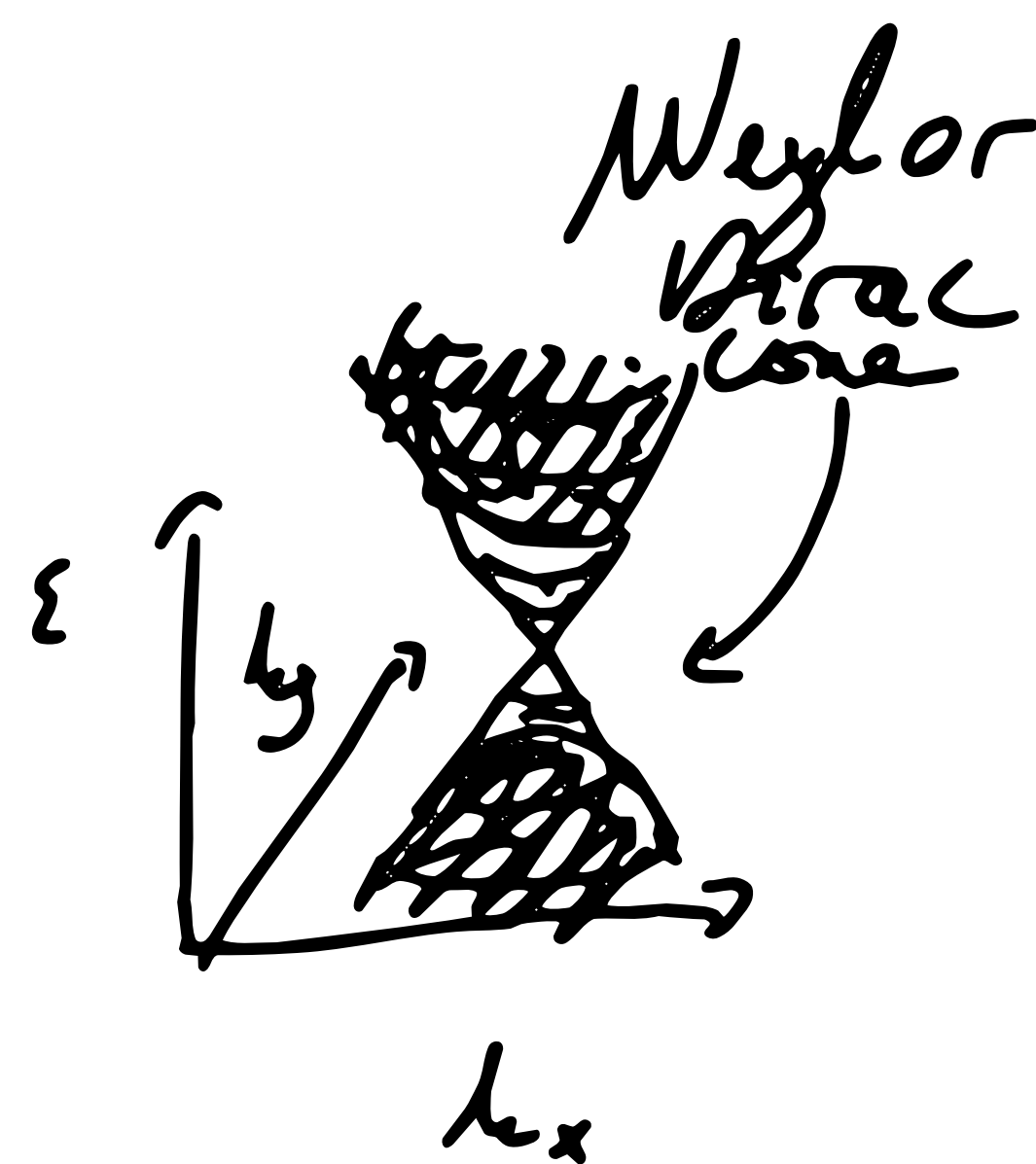
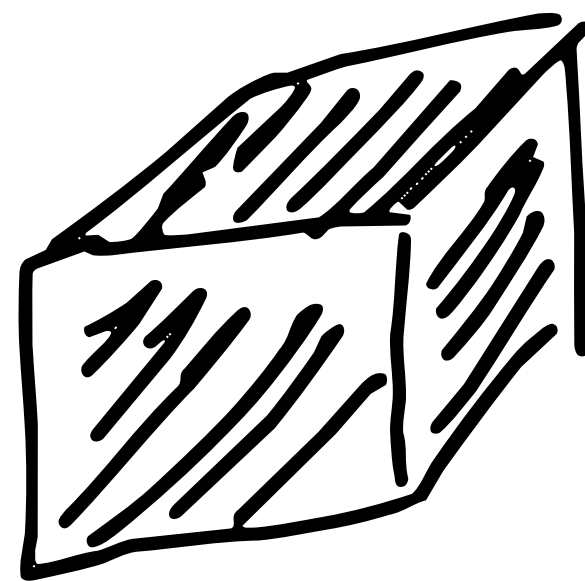
Majorana edge modes, 1D
(localized nonabelian anyons)



3D example

Bi₂Te₃

Fe₂Te₃



Phenomenology:

- Single edge channels contribute with one quantum of conductance

$$R_K = \frac{e^2}{h} = 25,812. \dots \Omega$$

- localized edge states (exponentially)
- immunity against disorder & gapping (until both properties destroyed)

2. Topological classification of noninteracting electronic matter

2.1. Wigner's theorem 1931

Symmetries of nature are incorporated
in quantum mechanics by unitary,
linear operators or antiunitary,
anti-linear operators without exceptions.
(determined up to a phase)

Examples: Rotation symmetry, translation
symmetry, crystallographic
symmetries $\rightarrow$ unitary

time-reversal & particle-hole
symmetry (charge conjugation) $\rightarrow$ anti-
unitary

For states ψ_1 & ψ_2 and scalars α & β

$$\begin{array}{cc} \text{linear} & \text{antilinear} \\ U(\alpha\psi_1 + \beta\psi_2) = \alpha U\psi_1 + \beta U\psi_2 & U(\alpha\psi_1 + \beta\psi_2) = \alpha^* U\psi_1 + \beta^* U\psi_2 \end{array}$$

$$\begin{array}{cc} \text{unitary} & \text{antiunitary} \\ (U\psi_1, U\psi_2) = (\psi_1, \psi_2) & (U\psi_1, U\psi_2) = (\psi_1, \psi_2)^* \end{array}$$

Alternatively, choose basis ψ_j & define complex conjugation operator K

$$K\psi = K\left(\sum_j \alpha_j \psi_j\right) = \sum_j \alpha_j^* \psi_j$$

Every antiunitary operator is uniquely written as the product of a unitary operator & complex conjugation

$$\begin{array}{ccc} & U = A \cdot K & \\ \text{antiunitary} \swarrow & \downarrow & \searrow \text{conjugation} \\ & \text{unitary} & \end{array}$$

2.2. Time-reversal, particle-hole, & chiral symmetry

Consider family of Hamiltonians $\mathcal{H}(p)$, where p are the system parameters.

An operator U can fundamentally have the following properties

$$[U, \mathcal{H}] = 0 \quad U \text{ unitary} \Rightarrow \text{(unitary) symmetry}$$

$$[U, \mathcal{H}] = 0 \quad U \text{ antiunitary} \Rightarrow \text{time-reversal symmetry}$$

$$\{U, \mathcal{H}\} = 0 \quad U \text{ antiunitary} \Rightarrow \text{particle-hole symmetry}$$

$$\{U, \mathcal{H}\} = 0 \quad U \text{ unitary} \Rightarrow \text{chiral symmetry}$$

Recipe for classifying top. matter

- 1) Find out all unitary symmetries, block diagonalize the family of Hamiltonians. Complete set of pairwise commuting symmetry operators U_j .
Find simultaneous eigenvectors ψ_j & diagonalizing matrices $A = (\psi_1^T, \psi_2^T, \dots, \psi_n^T)$.
 $\mathcal{H}_D = A^\dagger \mathcal{H} A$ is block diagonal

$$H_{0,ij} = \begin{pmatrix} H_{0,1} & 0 & 0 \\ 0 & H_{0,2} & 0 \\ 0 & 0 & \ddots \end{pmatrix}_{ij}$$

Trick: Apply the diagonalizing transformation of random representatives to other random representatives $\Rightarrow$ only symmetry blocks remain

2) For each $H_{0,i}$ block, analyze the fundamental symmetries

time-reversal: $[T, H] = 0$, T anti-unitary

physical candidates:

• spinless systems $T = K$, $T^2 = 1$

• spinful systems $T C_{\uparrow} T^{-1} = C_{\downarrow}$
 $T C_{\downarrow} T^{-1} = -C_{\uparrow}$, $T^2 = -1$

• otherwise generalized time-reversal symm.
 $T^2 = \pm 1$

phenomena: Kramers' degeneracy (for $T^2 = -1$),
 magnetic field break (physical)
 TRS of spinful fermions.

particle-hole (charge conjugation)
 $\{P, H\} = 0$, P antiunitary

candidates

plain particle-hole symmetry
 $P C_{\sigma}^{\dagger} P^{-1} = C_{\sigma}$ $P^2 = 1$

orbital mixing
 $P C_{\sigma}^{\dagger} P^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} C_{\sigma'}$ $P^2 = \pm 1$

phenomena: Symmetric spectrum, (attention, also for chiral symmetry)
typical for Bogoliubov-de-Gennes formalism in superconductors.

Chiral symmetry

$\{C, H\} = 0$ C unitary

candidates:

- $T \cdot P$ (works if T & P are time-reversal & particle-hole symmetry, respectively)

- unitary part of antiunitary symmetries

phenomena: Chirally invariant, degenerate null space

- symmetric spectrum (but no momentum inversion, no conj.)

Properties of T, P, C

$$\left. \begin{aligned} TP &= C \\ PC &= T \\ TC &= P \end{aligned} \right\} \text{up to phase}$$

by choice of phase

$$T^2 \in \{-1, 1\}; \quad P^2 \in \{-1, 1\}; \quad C^2 \stackrel{!}{=} 1$$

T, P & C are unique - up to phase - because there is no unitary symmetry left.

Example proof for time-reversal theory

T & T' are time reversal symmetries.

$$\text{Then } [TT', \mathcal{H}] = TT'\mathcal{H} - \mathcal{H}TT' \stackrel{\text{also}}{=} 0.$$

TT' is unitary:

$$(TT'\psi_1, TT'\psi_2) = (T'\psi_1, T'\psi_2)^* = (\psi_1, \psi_2)$$

& TT' is linear

$$\begin{aligned} TT'(\alpha_1\psi_1 + \alpha_2\psi_2) &= T(\alpha_1^* T'\psi_1 + \alpha_2^* T'\psi_2) \\ &= \alpha_1 TT'\psi_1 + \alpha_2 TT'\psi_2. \end{aligned}$$

$\Rightarrow TT'$ is a unitary symmetry

$$\Rightarrow TT' \propto 1.$$

$$\Rightarrow T' \propto T^{-1} \propto T$$

$\hat{=}$ $T^2 = \pm 1$

Example proof for $T^2 = \pm \underline{1}$

T^2 unitary & linear & symmetry, as before
 $\Rightarrow \text{Tr}\{T^2\} = e^{i\varphi} \underline{1}$

$$T = AK$$

$$\Rightarrow T^2 = AA^*$$

cyclic
trace property

$$\text{Tr}\{T^2\}^* = \text{Tr}\{AA^*\}^* = \text{Tr}\{A^*A\} \stackrel{\text{cyclic trace property}}{=} \text{Tr}\{AA^*\} = \text{Tr}\{T^2\}$$

$$\Rightarrow e^{-i\varphi} = e^{i\varphi} \Rightarrow \varphi = \pi \cdot n, \quad n \text{ integer}$$

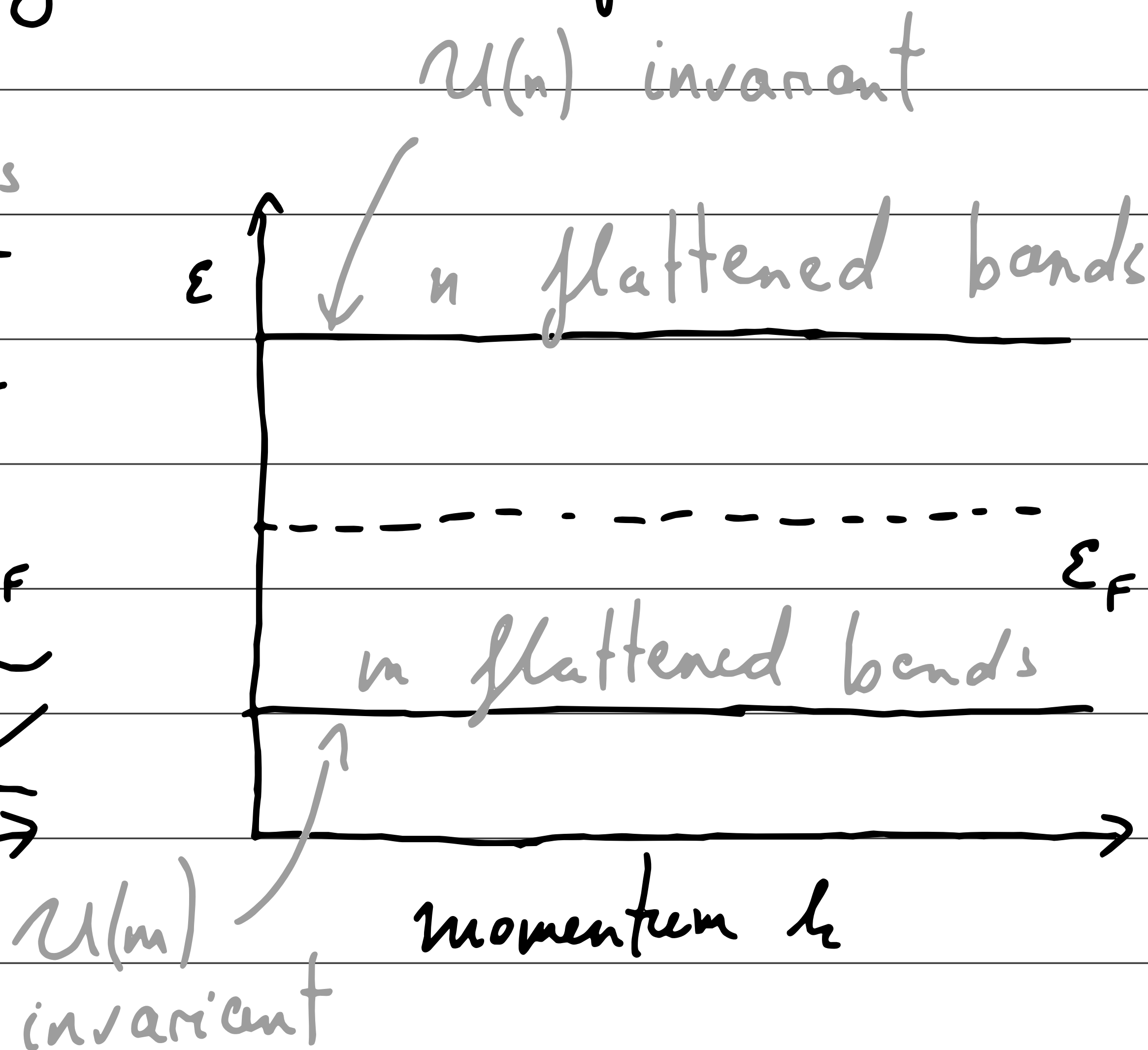
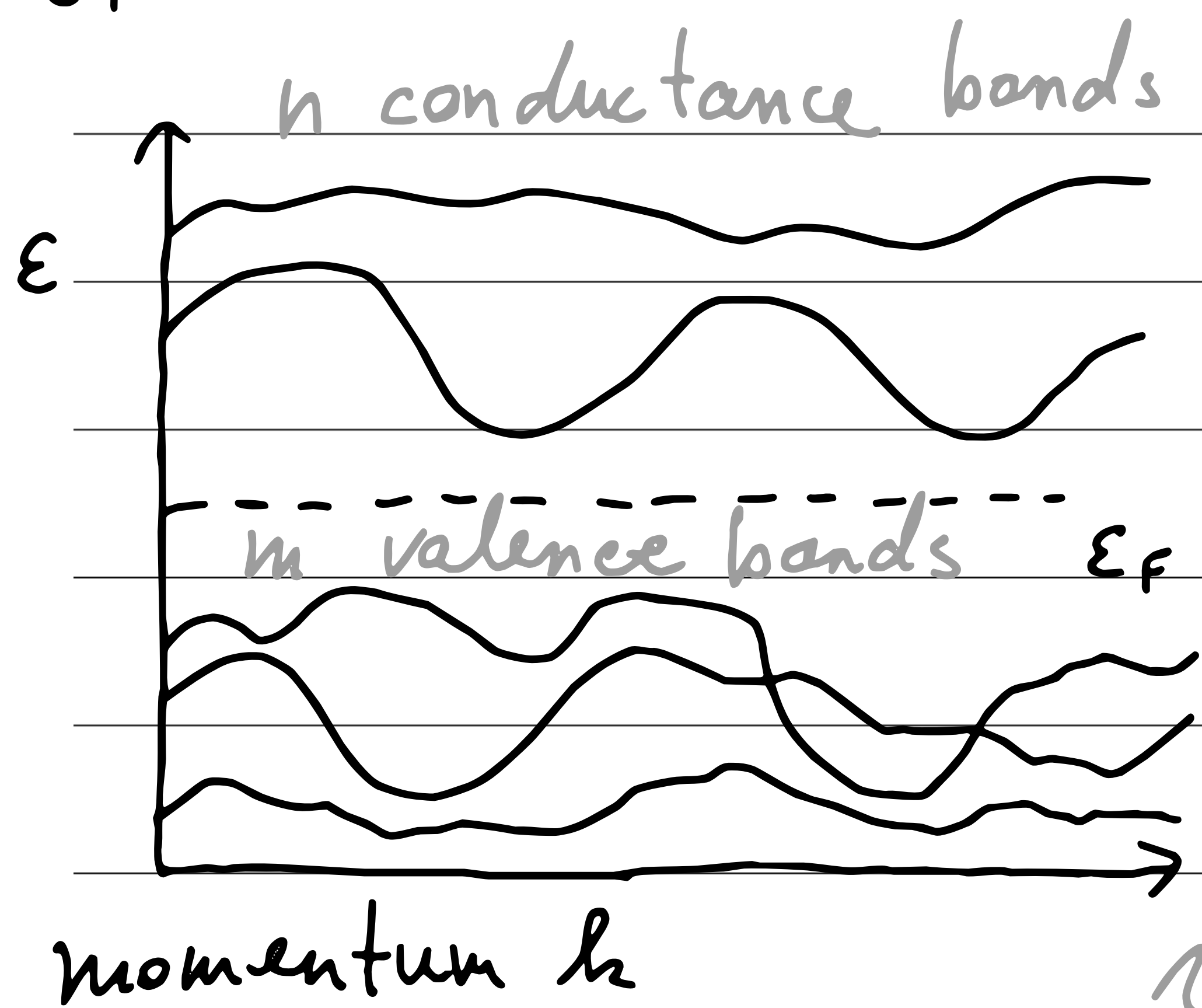
$$\Rightarrow T^2 = \pm \underline{1}!$$

23. Topological invariants (basics)

$$H = \sum_{j,l} A_{jl} c_j^\dagger c_l + B_{jl} c_j^\dagger c_l^\dagger + B_{jl}^* c_l c_j$$

Derive flat band Hamiltonian as most accessible representative of the symmetry class. For simplicity, no symmetry assumed here (no T, P, C symmetry), symmetry class A .

$Q = P_{\text{unocc.}} - P_{\text{occ.}}$ with $P_{\text{occ.}} + P_{\text{unocc.}} = \mathbb{1}$, $P^\dagger = P$, $P^2 = P$.
 projector to unoccupied states $\leftarrow$ projector to occupied state $\leftarrow$ $U(n)$ invariant



$$\text{We find } QQ^\dagger = (P_u - P_o)^2 = P_u - \underbrace{P_o P_u - P_u P_o}_{=0} + P_o = \underline{1}.$$

This means Q is unitary, its valence & conduction bands can be rotated among the same kind without changing Q

$$Q_k \in U(n+m) / (U(n) \times U(m)) = Gr(n+m, n).$$

We are interested in physics that is independent of the number of bands. (Also, number of conduction bands is infinite)

Band crossings of valence & conduction bands $\Rightarrow$ topological phase transitions would cause

$$Q: T^n \rightarrow Gr(m, n+m) \text{ discontinuous}$$

$\hookrightarrow$ Brillouin zone
n-dimensional
torus

[Toric homotopy
group: Fox 1948]

$\Rightarrow$

toric homotopy group of maps $\varphi: T^n \rightarrow Gr(n+m, n)$ give the topological classes & their $\mathbb{Z}_2$ representation the topological invariant.

$\hookrightarrow$ Look up in periodic table

[Kitaev 2003; Ryu, Schnyder, Furusaki, Ludwig 2009, 2010]
(K-theory) (nonlinear sigma model & Anderson localization)

Important classification theorem
for torus homotopy groups

→ dimension of system

$$\tau_d \approx \prod_{j=1}^d \binom{d}{j} \pi_j \quad [\text{Fox 1948}]$$

strictly
correct

notation

$$\left[\tau_d \approx \prod_{j=1}^d \pi_j \oplus \binom{d}{j} \right]$$

j^{th} homotopy group of
spheres

$S^d \rightarrow$ target manifold.

Very physical interpretation: (see graphics
in two pages)

π_1 : strong invariant, remains valid
if crystal symmetry (Brillouin zone
picture) is broken

$\pi_{j < d}$: $\binom{d}{j}$ different π_j 'weak' topological
invariants (broken if crystal symm.
fails), exactly one for each direction
like system constructed by stacked
lower-dimension ones.

Full description

e.g. $d=3$

index: $\left(\binom{d=3}{1}, \binom{d=3}{2}, \binom{d=3}{3}, \binom{d=3}{0}, \binom{d=2,xy}{1}, \binom{d=2,xy}{2}, \binom{d=2,xy}{0}, \binom{d=2,yz}{1}, \binom{d=2,yz}{2}, \binom{d=2,yz}{0}, \binom{d=2,zx}{1}, \binom{d=2,zx}{2}, \binom{d=2,zx}{0}, \binom{d=1,x}{1}, \binom{d=1,x}{0}, \binom{d=1,y}{1}, \binom{d=1,y}{0}, \binom{d=1,z}{1}, \binom{d=1,z}{0}, \binom{d=0}{0} \right)$

2.4 The tenfold way

Cartan
[1928]

With $T^2, P^2 = \pm 1$ & $C = T \cdot P$,

we have 10 different possibilities

Cartan
label

T^2 P^2 C^2 (target dimension) complex Bott periodic.

	T^2	P^2	C^2	(target manifold)	1	2	3	4	5
A	0	0	0	$Gr(n, N)$	/	$\mathbb{Z}_2$	/	$\mathbb{Z}_2$	/
$AIII$	0	0	1	$U(\frac{N}{2})$	$\mathbb{Z}_2$	/	$\mathbb{Z}_2$	/	$\mathbb{Z}_2$
AI	1	0	0	$\frac{O(N)}{O(n) \times O(m)}$	/	/	/	$\mathbb{Z}$	/
BDI	0	1	1	$O(N)$	$\mathbb{Z}$	/	/	/	$\mathbb{Z}$
D	-1	1	0	$O(2N)/U(N)$	$\mathbb{Z}_2$	$\mathbb{Z}$	/	/	/
$DIII$	-1	1	1	$U(2N)/Sp(N)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	/	/
$ATII$	-1	0	0	$Sp(N)/Sp(n) \times Sp(m)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	/
CII	0	-1	1	$Sp(N)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
C	1	-1	0	$Sp(2N)/U(N)$	/	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
CII	1	-1	1	$U(N)/O(N)$	/	/	$\mathbb{Z}$	/	$\mathbb{Z}_2$

exercise:

$N = n + m$ = total number of bands; doubled for superconductors ($2N$)

Show these are all possible classes more systematically

Cartan 1926, 1927

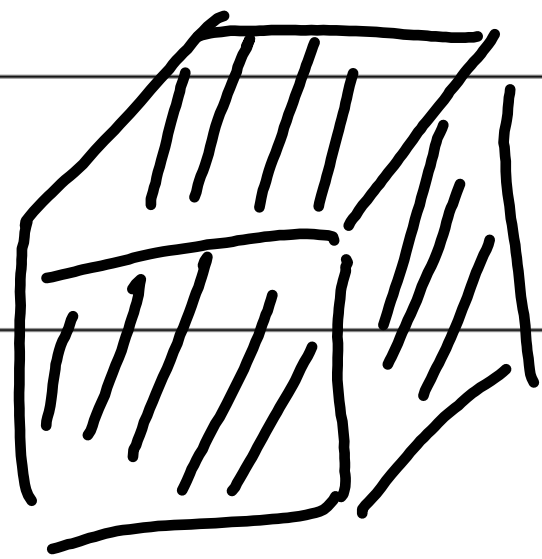
Altland & Zirnbauer 1996, '97

Kitaev 2009

Ryu, Schnyder, Furusaki, Ludwig '09, '10.

Interesting: There are more symmetric spaces but only 10 of them can be picked because of the limited T, P & C combinations

Example: Dimension $d=3$, class $\mathbb{O}III$



strong topological
index $d=3$

weak indices



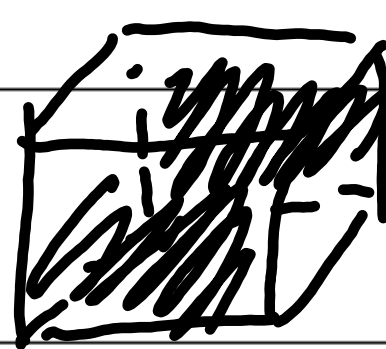
$d=2, xy$



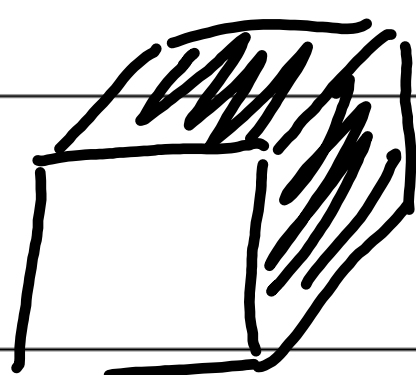
$d=1, x$



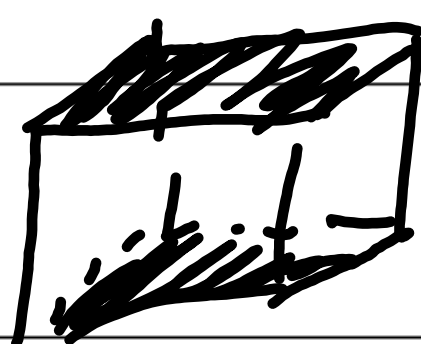
$d=2, yz$



$d=1, y$



$d=2, zx$



$d=1, z$

$d=0$



3. Majoranas - fundamentals

Consider a fermionic mode

① annihilation, creation operators
 $c, c^\dagger, n = c^\dagger c$ (counts occupation)

Introduce real fermions (Majorana 1932, '37)

② $\chi^1 = c + c^\dagger$
 $\chi^2 = (c - c^\dagger)/i$ $\chi^{j\dagger} = \chi^j$ $i\chi^\mu \partial_\mu \chi - m\chi^2 = 0$
↳ different
Real fields

These still have fermion-like anticommutations

$$\{\chi^1, \chi^2\} = \frac{1}{i} \underbrace{\{c, c^\dagger\}}_{=1} + \frac{1}{i} \underbrace{\{c^\dagger, c\}}_{=1} = 0$$

$$\{\chi^1, \chi^1\} = c c^\dagger + c^\dagger c = 1$$

$$\{\chi^2, \chi^2\} = + (c c^\dagger + c^\dagger c) = 1$$

$$\text{But } \chi^{1\dagger} \chi^1 - \chi^{12} = \frac{1}{2} \{\chi^1, \chi^1\} = \frac{1}{2}$$

! "One cannot count occupation of a Majorana mode"

Question:

If every fermion consists of two Majoranas, , why are Majoranas special then? Shouldn't they be observable everywhere?

Answer: The Majoranas are typically bound in pairs at the same location. Special setups are necessary to separate them.

3.1. The Kitaev Majorana chain

Model: Spinless fermions on 1d superconducting lattice [Kitaev 2001]

$-W c_{j+1}^\dagger c_j$ $\Delta c_{j+1}^\dagger c_j^\dagger$ $\mu c_j^\dagger c_j$ $\{c_j, c_{j'}\} = 0$

[Lieb, Schultz, Mattis '61] [Pfeuty '70] found edge modes
 (also constructed Majorana operators)

$\{c_j, c_{j'}^\dagger\} = \delta_{jj'}$ (interesting: Not [Haldane 1983])

$\mathcal{H}(W, \Delta, \mu) = \sum_{j=1}^{N-1} -W c_{j+1}^\dagger c_j + \Delta c_{j+1}^\dagger c_j^\dagger + h.c. - \sum_{j=1}^N \mu (c_j^\dagger c_j - \frac{1}{2})$

(gauge away $c_j \rightarrow e^{i\varphi_j} c_j$ except on S^1)

$= \frac{1}{2} \sum_{j,j'=1}^N (c_j^\dagger, c_j) \begin{pmatrix} -W(\delta_{j,j'+1} + \delta_{j+1,j'}) - \mu \delta_{j,j'+1} & 2\Delta \delta_{j,j'+1} \\ 2\Delta \delta_{j+1,j'} & W(\delta_{j,j'+1} + \delta_{j+1,j'}) + \mu \delta_{j,j'} \end{pmatrix} \begin{pmatrix} c_{j'}^\dagger \\ c_{j'} \end{pmatrix}$

Introduce two Majorana operators per site

$\gamma_j^1 = c_j + c_j^\dagger$ $\gamma_j^2 = (c_j - c_j^\dagger) \frac{1}{i}$

These are real operators & fulfill fermionic relations

$\{\gamma_j^1, \gamma_{j'}^2\} = \overbrace{\{c_j, c_{j'}^\dagger\}}^{\delta_{jj'}} \frac{-1}{i} + \overbrace{\{c_{j'}, c_j^\dagger\}}^{\delta_{jj'}} \frac{1}{i} = 0$

$\{\gamma_j^1, \gamma_{j'}^1\} = \{c_j, c_{j'}^\dagger\} + \{c_{j'}, c_j^\dagger\} = 2 \delta_{jj'}$

$\{\gamma_j^2, \gamma_{j'}^2\} = \{c_j, c_{j'}^\dagger\} (\frac{1}{i})^2 (-1) + \{c_{j'}, c_j^\dagger\} (\frac{1}{i})^2 (-1) = 0$

$\{\gamma_j^1, \gamma_{j'}^1\} = 2 \delta_{jj'}$

Reformulate the Hamiltonian in Majorana operators $(\gamma_j^\dagger) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} (c_j)$

$$\begin{pmatrix} \gamma_j^1 \\ \gamma_j^2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}}_{\text{unitary}} \begin{pmatrix} c_j \\ c_j^\dagger \end{pmatrix}$$

$$uu^t = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2 \cdot \underline{\underline{1}} \quad u \Rightarrow u^{-1} = u^t/2.$$

$$\Rightarrow \mathcal{H} = \sum_{j,j'} (\gamma_j^1, \gamma_j^2) u_h u^\dagger \begin{pmatrix} \gamma_{j'}^1 \\ \gamma_{j'}^2 \end{pmatrix} \cdot \frac{1}{8}.$$

$$\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} a+b & i(a-b) \\ c-a & i(c+a) \end{pmatrix} = \begin{pmatrix} b+c & i(2a+c-b) \\ -i(2a+b-c) & -c-b \end{pmatrix}$$

$$= \frac{i}{2} \sum_{j, j'} (f_j^1, f_j^2) h_\mu \begin{pmatrix} f_{j'}^1 \\ f_{j'}^2 \end{pmatrix} + [(b+c) + (-c-b)] / 8$$

with $h_\mu = \begin{pmatrix} 0 & -\omega(\delta_{jj'+1} + \delta_{j+1,j'}) - \mu \delta_{jj'} + \Delta(\delta_{j+1,j'} - \delta_{jj'+1}) \\ \omega(\delta_{jj'+1} + \delta_{j+1,j'}) + \mu \delta_{jj'} - \Delta(\delta_{j+1,j'} - \delta_{jj'+1}) & 0 \end{pmatrix}$ (diagonal terms from $\gamma_j^2 = \gamma_{j'}^2 = 1$)

$$a = -w(\delta_{j,j'+1} + \delta_{j+1,j'}) - \mu \delta_{jj'}$$

$$b = 2 \Delta \int_{j, j'+1}$$

$$c = 2 \Delta \delta_{jx, j'}$$

Alternative calculation without matrices

$$-\omega c_{j+1}^+ c_j + \text{h.c.} = -\frac{\omega}{4} \left[\gamma_{j+1}^1 \gamma_j^1 + \gamma_{j+1}^2 \gamma_j^2 + i (\gamma_{j+1}^2 \gamma_j^1 + \gamma_{j+1}^1 \gamma_j^2) \right] + \text{h.c.}$$

$$= i \frac{\omega}{2} (-\gamma_{j+1}^1 \gamma_j^2 + \gamma_{j+1}^2 \gamma_j^1).$$

$$-\mu c_j^+ c_j = i \frac{\mu}{2} (\gamma_j^1 \gamma_j^2 - \gamma_j^2 \gamma_j^1 + 2) = -i \frac{\mu}{2} \gamma_j^1 \gamma_j^2 + \frac{-i\mu}{2}$$

$$\Delta c_{j+1}^+ c_j^+ + \text{h.c.} = -\frac{\Delta}{4} \left[i (\gamma_{j+1}^2 \gamma_j^1 + \gamma_{j+1}^1 \gamma_j^2) \right] + \text{h.c.}$$

$$= -\frac{i\Delta}{2} (\gamma_{j+1}^2 \gamma_j^1 + \gamma_{j+1}^1 \gamma_j^2).$$

$$\Rightarrow \mathcal{H} = \sum_{j=1}^{N-1} -\omega c_{j+1}^+ c_j + \Delta c_{j+1}^+ c_j + \sum_{j=1}^N -\mu (c_j^+ c_j - \frac{1}{2})$$

↪

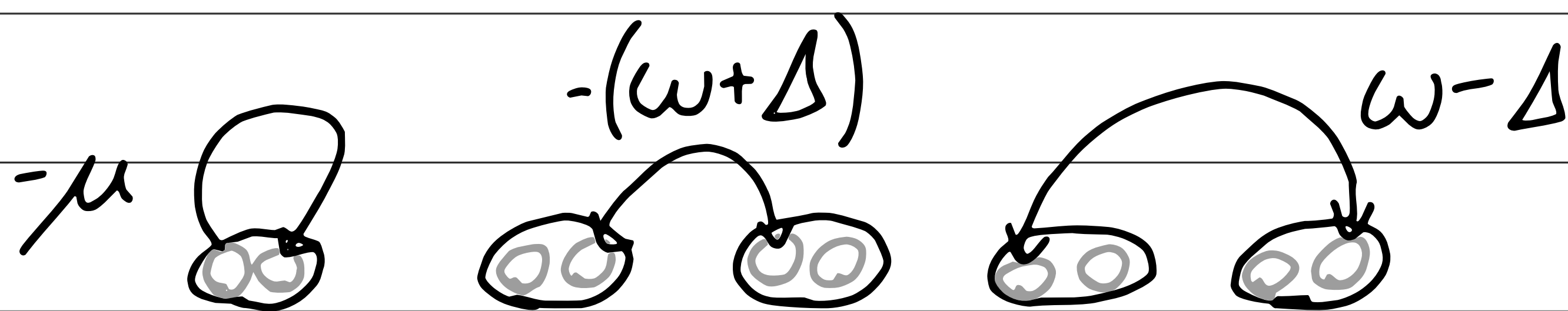
$$\mathcal{H} = \frac{i}{2} \sum_{j=1}^{N-1} -\gamma_{j+1}^1 \gamma_j^2 (\omega + \Delta) + \gamma_{j+1}^2 \gamma_j^1 (\omega - \Delta) + \frac{i}{2} \sum_{j=1}^N -\mu \gamma_j^1 \gamma_j^2.$$

(v)
equivalent
to Kitaev's
manuscript.

$$\begin{aligned} \Rightarrow \mathcal{H} &= \frac{i}{4} \sum_{j=1}^{N-1} -w (\overbrace{\gamma_j^1 \gamma_{j+1}^2}^{\gamma_{j+1}^2 \gamma_j^1} + \gamma_{j+1}^1 \gamma_j^2) + \mu \gamma_j^1 \gamma_j^2 + \approx \\ &\approx \Delta (\overbrace{\gamma_j^2 \gamma_{j+1}^1}^{\gamma_{j+1}^1 \gamma_j^2} - \gamma_{j+1}^1 \gamma_j^2) + h.c. \end{aligned}$$

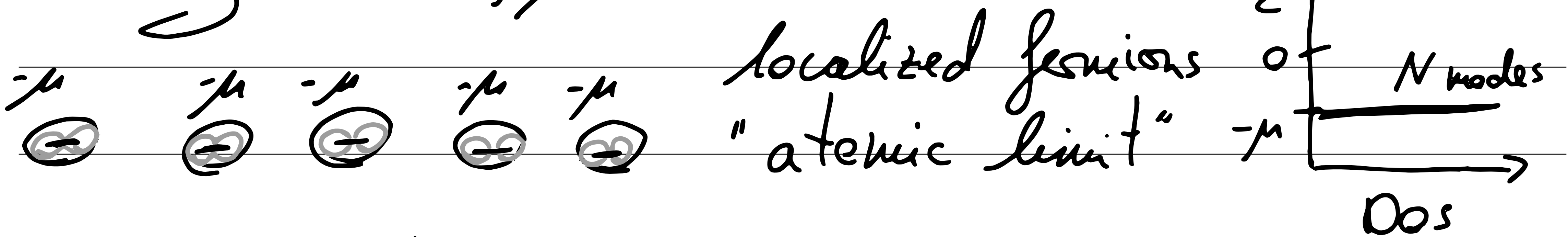
$$= \frac{i}{4} \sum_{j=1}^{N-1} -\mu \gamma_j^1 \gamma_j^2 - (\omega + \Delta) \gamma_{j+1}^1 \gamma_j^2 + (\omega - \Delta) \gamma_{j+1}^2 \gamma_j^1 + h.c.$$

We reach at the Majorana representation of the Kitaev Majorana chain

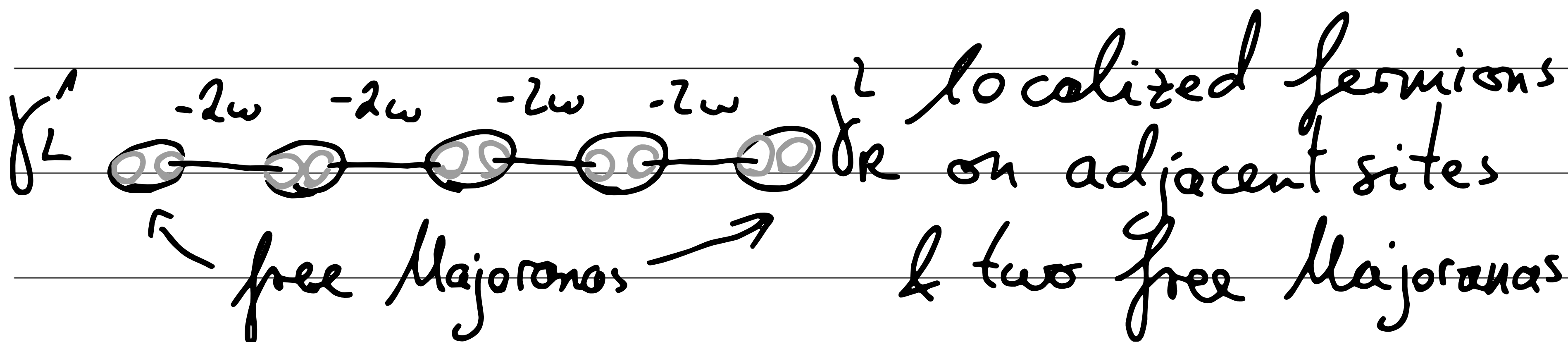


Interestingly some connections are missing $\gamma_j^1 \gamma_{j+1}^1$ $\gamma_j^2 \gamma_{j+1}^2$ ($w, \Delta \in \mathbb{R}$)

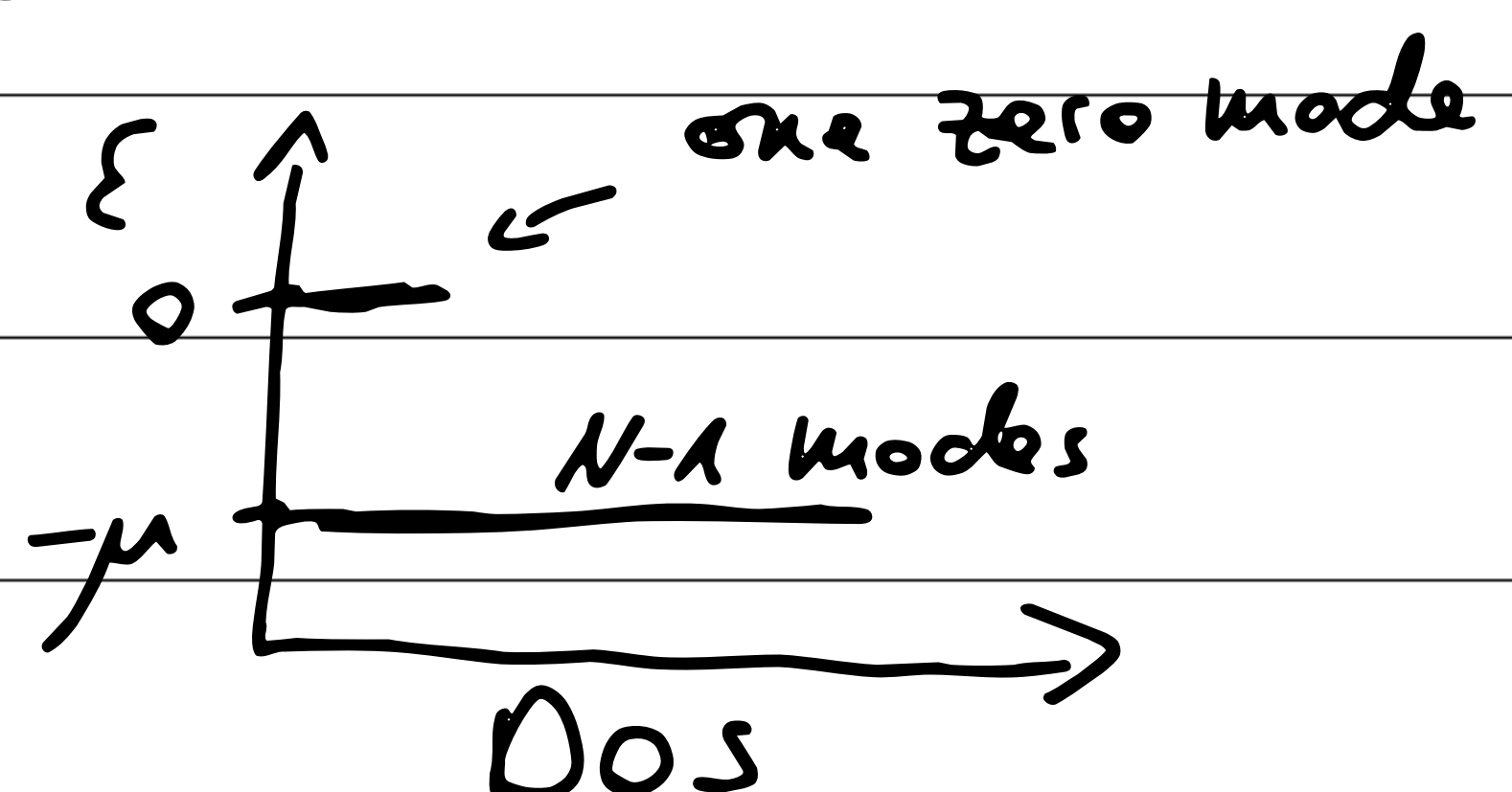
limiting cases a) $\mu \neq 0, \omega = \Delta = 0$



b) $\mu = 0, \omega = -\Delta \neq 0$



Form protected zero mode
 $c_0 = (\gamma_L^1 + i \gamma_R^2) / 2$



Symmetry analyzes of the Kitaev Maj. chain

particle hole symmetry

$$P \alpha c^\dagger P^{-1} = \alpha^* c$$

$$P \mathcal{H} P^{-1} = \sum_{j=1}^N \underbrace{-\omega}_{\text{real}} P \left(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1} \right) P^{-1}$$

$$+ \underbrace{\Delta}_{\text{real}} P \left(c_{j+1}^\dagger c_j^\dagger + c_j c_{j+1} \right) P^{-1}$$

$$\sum_{j=1}^N -\mu P \left(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1} \right) P^{-1}$$

$$c_j c_j^\dagger - \frac{1}{2} = -c_j^\dagger c_j + 1 - \frac{1}{2} = -\left(c_j^\dagger c_j - \frac{1}{2} \right).$$

$$= -\mathcal{H} \Rightarrow \{P, \mathcal{H}\} = 0 \quad \checkmark$$

The Hamiltonian has a particle-hole symmetry

time reversal symmetry

No spin. Candidate $T = K$

$THT^{-1} = H$ because $\omega, \Delta, \mu \in \mathbb{R}$.

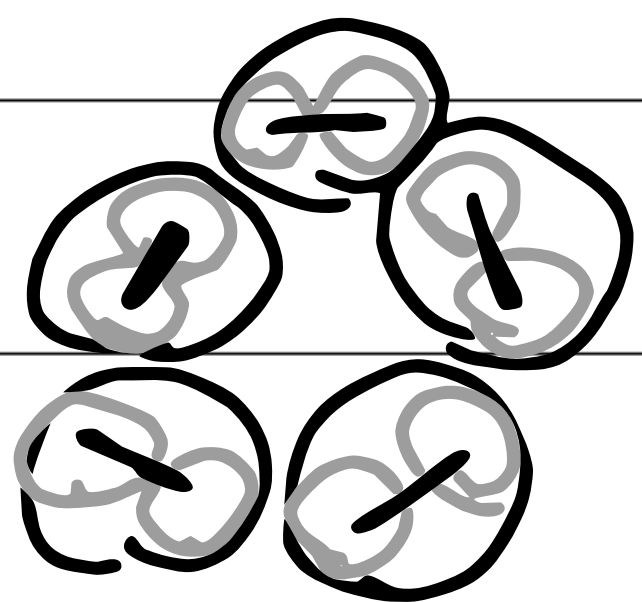
However, any fluctuation in the superconducting phase

$$H_F = \eta \left(c_{j+1}^\dagger c_j^\dagger + \eta^* c_j c_{j+1} \right)$$

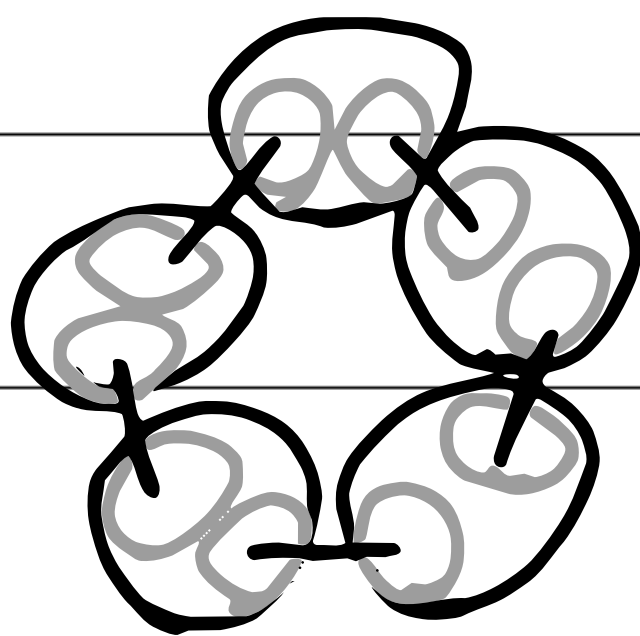
with complex η would break time reversal symmetry
 $\Rightarrow$ unstable

3.2. Topological invariant of the Majorana chain

What distinguishes the two Majorana phases at periodic boundary conditions?



$$Z = 1$$



$$Z = -1$$

Both carry only bound modes.

Kitaev 2001: Different ground state fermionic parity
 ↳ see there

$$Z = (-1)^{\sum_{j=1}^N n_j}$$

For $\mathcal{H} = \frac{i}{2} \sum_{j,j'} \gamma_j A_{jj'} \gamma_{j'}$,

the parity is the sign of the Pfaffian of the skew symmetry matrix $\underline{A}$.

$$Z = \text{sgn}(\text{Pf}(\underline{A}))$$

Background Pfaffian of a skew symmetric $2N \times 2N$ matrix half as many as for determinant

$$Pf(A) = \frac{1}{2^N N!} \sum_{\sigma \in \text{permutations}} \text{sgn}(\sigma) A_{\sigma(1), \sigma(2)} A_{\sigma(3), \sigma(4)} \cdots A_{\sigma(2N-1), \sigma(2N)}$$

For real, skew symmetric matrices, the Pfaffian & the determinant are the only $SO(2N)$ invariant polynomials (invariant against rotation)

Useful identities

$$Pf(A)^2 = \text{Det}(A)$$

$$Pf(B A B^T) = \text{Det}(B) Pf(A)$$

$$Pf(A_1 \oplus A_2) = Pf(A_1) Pf(A_2)$$

Example

$n=1$

$$Pf \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = \frac{1}{2} \left(\overset{+}{\underset{1\ 2}{\overset{1\ 2}{A_{12}}}} + \overset{-}{\underset{2\ 1}{\overset{1\ 2}{A_{21}}}} \right) = \underline{\underline{A_{12}}}$$

Matrix A consists of blocks (as derived) specified by their distance x from the diagonal

periodic

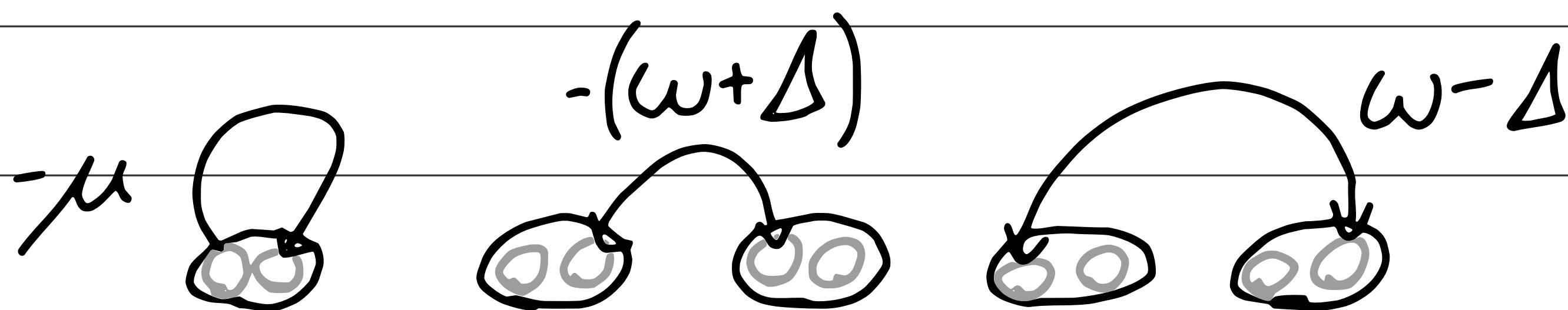
$$A = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \\ & & & \ddots & \ddots \\ & & & & \ddots \end{pmatrix}$$

$\dots \quad x_i^1 \quad x_i^2 \quad x_{i+1}^1 \quad x_{i+1}^2$

$x=0 \quad x=-1$

$x=0 \quad x=1$

Attention:
Not exactly
the same as
 h_n because
 j, j' commutators
change a sign



$$A_{\text{block}}(x) = \begin{pmatrix} 0 & -\mu \delta_{x,0} & -\omega(\delta_{x,1} + \delta_{x,-1}) + \Delta(\delta_{x,-1} - \delta_{x,1}) \\ \mu \delta_{x,0} + \omega(\delta_{x,1} + \delta_{x,-1}) + \Delta(\delta_{x,-1} - \delta_{x,1}) & 0 & 0 \end{pmatrix}$$

Decouple by Fourier transform

$$\tilde{A}_{\text{block}}(k) = \frac{1}{N} \sum_{x=-\frac{N}{2}}^{\frac{N}{2}} e^{ikx} h_n(x) = \begin{pmatrix} 0 & -\mu - \omega(e^{ik} + e^{-ik}) + \Delta(e^{-ik} - e^{ik}) \\ \mu + \omega(e^{ik} + e^{-ik}) + \Delta(e^{-ik} - e^{ik}) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\mu - 2\omega \cos(k) + \Delta i \sin(k) \\ \mu + \omega \cos(k) + \Delta i \sin(k) & 0 \end{pmatrix}$$

$$\tilde{A}_{\text{block}}(0) = \begin{pmatrix} 0 & -\mu - 2\omega \\ \mu + 2\omega & 0 \end{pmatrix} ; \quad \tilde{h}_n(\pi) = \begin{pmatrix} 0 & -\mu + 2\omega \\ \mu - 2\omega & 0 \end{pmatrix}$$

Remark: $\text{Det}(\tilde{A}_{\text{block}}(k)) = -\varepsilon(k)^2 < 0$
 with spectrum $\varepsilon(k)$

$$\Rightarrow \varepsilon(k) = \pm \sqrt{4\Delta \sin^2(k) + (\mu + 2w \cos(k))^2}$$

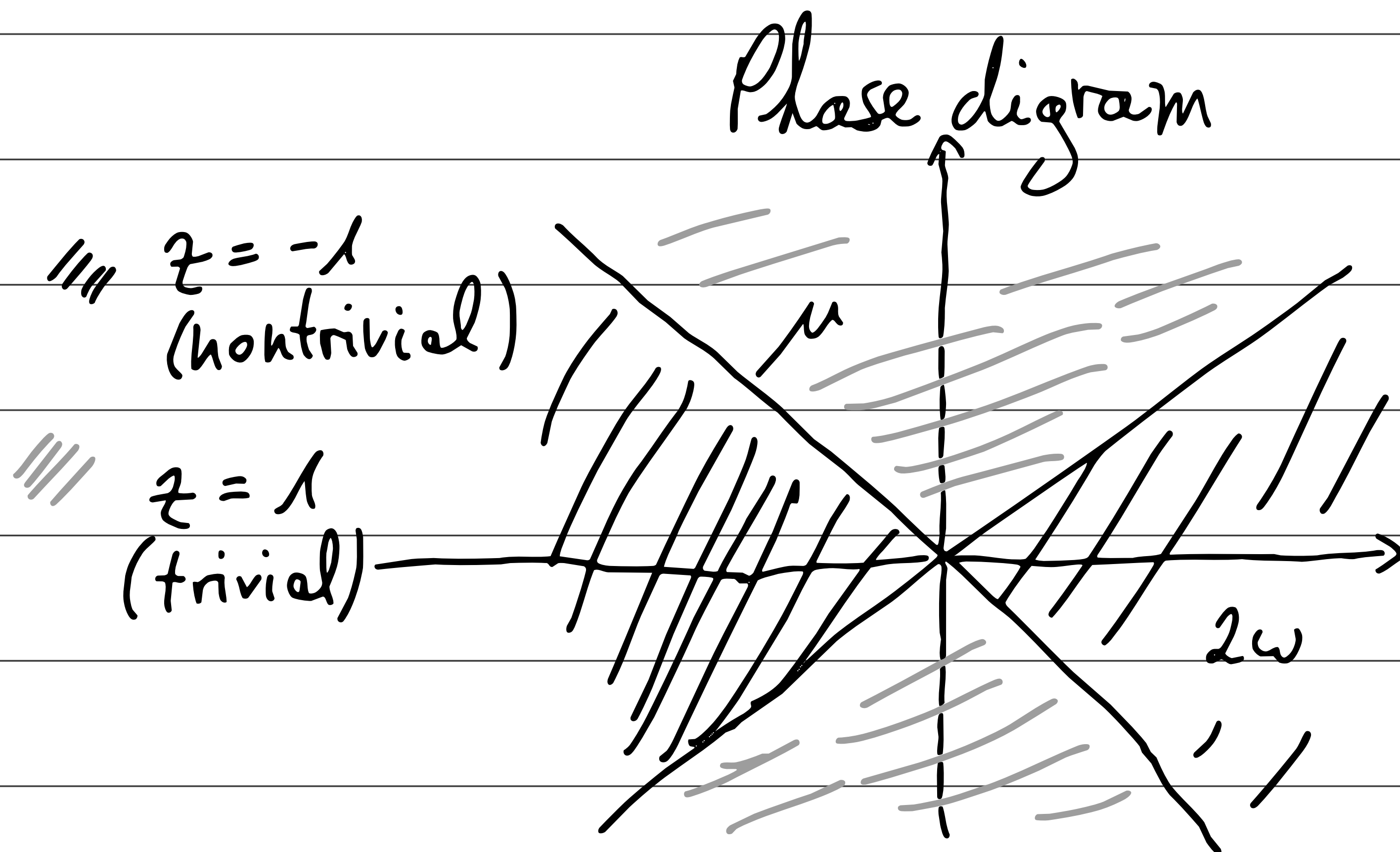
Use [exercise sheet]

$$\text{sgn}[\text{Pf}(A)] = \text{sgn} \left[\underbrace{\prod_{k \neq 0, \pi} \det(\tilde{A}_{\text{block}}(k))}_{\text{pair } k \& -k \Rightarrow > 0} \cdot \text{Pf}(\tilde{A}_{\text{block}}(0)) \cdot \text{Pf}(\tilde{A}_{\text{block}}(\pi)) \right]$$

$$\begin{aligned} \text{Pf}(\tilde{A}_{\mu}(0)) &= \text{Pf} \begin{pmatrix} 0 & -\mu - 2w \\ \mu + 2w & 0 \end{pmatrix} = -(\mu - 2w) \\ \text{Pf}(\tilde{A}_{\mu}(\pi)) &= \text{Pf} \begin{pmatrix} 0 & -\mu + 2w \\ \mu - 2w & 0 \end{pmatrix} = -\mu + 2w \end{aligned}$$

$$\Rightarrow Z = \text{sgn}(\text{Pf}(A)) = \text{sgn}(\mu^2 - 4w^2)$$

$$\Rightarrow Z = \text{sgn}(|\mu| - |2w|)$$



$\Delta \neq 0$ does
 not change
 phase diagram

