

RIKEN Majorana Workshop

exercise sheet 1

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1.1 Calculation with Pfaffians

In this exercise, we derive a formula for the Pfaffian of a $2n \times 2n$ skew symmetric matrix A , the entries of which only depend on their offset from the diagonal:

$$A_{i,j} = A(j-i). \quad (1)$$

Here, $j-i$ has to be seen modulo $2n$. This formula is used in the lecture to derive the $\mathbb{Z}_2$ invariant of the Kitaev chain.

Hint: The following formulas may be useful

$$\text{Pf}(A)^2 = \det(A), \quad (2)$$

$$\text{Pf}(BAB^T) = \text{Pf}(A) \det(B), \quad (3)$$

$$\text{Pf}(A_1 \oplus A_2) = \text{Pf}(A_1) \text{Pf}(A_2). \quad (4)$$

- a. (1 point) The Fourier transform of such a matrix is defined as $\tilde{A}(q) = \sum_j e^{iqj} A(j)$ with $q \in BZ = \frac{2\pi}{n} \{0, \dots, n-1\}$. On the other hand $U_{q,j} = \frac{1}{\sqrt{n}} e^{iqj}$ is the Fourier transform matrix. Show that $(UAU^T)_{q_1, q_2} = \delta_{q_1, -q_2} \tilde{A}(q_2)$.

- b. (1 point) Derive

$$\text{Pf} \begin{pmatrix} 0 & M \\ -M^T & 0 \end{pmatrix} = s \det(M) \quad (5)$$

for an arbitrary matrix $n \times n$ matrix M , where $s \in \{-1, 1\}$ is just a sign depending on n .

- c. Now combine the previous results to derive

$$\text{Pf}(A) \propto \prod_{0 < q < \pi} \det(A(q)) \times \begin{cases} \prod_{q \in \{0, \pi\}} \text{Pf}(\tilde{A}(q)) & n \text{ even} \\ \text{Pf}(\tilde{A}(0)) & n \text{ odd.} \end{cases} \quad (6)$$

Hint: You need to reorder the submatrices of A in order to apply Eq. (4) and Eq. (5).

1.2 10-fold way

Given a Hamiltonian $\mathcal{H}$ that has no unitary symmetry, a time reversal operator $\mathcal{T}$ is an antiunitary operator that commutes with $\mathcal{H}$, a particle hole operator $\mathcal{P}$ is an antiunitary operator that anticommutes with $\mathcal{H}$ and a chiral symmetry $\mathcal{C}$ is a unitary operator that anticommutes with $\mathcal{H}$. These operators, if they exist, square to ± 1 , i.e., $\mathcal{T}^2, \mathcal{P}^2, \mathcal{C}^2 \in \{-1, 1\}$.

- a. (2 points) Show that $\mathcal{C} = \mathcal{T}\mathcal{P}$, $\mathcal{T} = \mathcal{C}\mathcal{P}$, $\mathcal{P} = \mathcal{C}\mathcal{T}$ (understood up to possible scalar phases).

- b. The 10-fold way says that there are exactly 10 possibilities for a given operator $\mathcal{H}$ to be classifiable by such operators $\mathcal{T}$, $\mathcal{P}$, and $\mathcal{C}$, considering the possibilities that $\mathcal{T}$, $\mathcal{P}$, and $\mathcal{C}$ either exist, or not exist and each squares either to $+1$ or -1 (independently of each other). *Hint: it is actually fair to assume $\mathcal{C}^2 = 1$. Why?* Make a table of these 10 possibilities, this table could for instance look like the one to the right, where 0 means that the respective operator does not exist and ± 1 means that the respective operator exists and squares to ± 1 , respectively.

$\mathcal{T}^2$	$\mathcal{P}^2$	$\mathcal{C}^2$
0	0	0
0	0	1
$\dots$	$\dots$	$\dots$

Kramer's degeneracy

Consider a Hamiltonian H with a time reversal symmetry T that squares to -1 , i.e., an antiunitary symmetry T with $[H, T] = 0$ and $T^2 = -1$.

- a. (2 points) Show that for each eigenvector $|\phi\rangle$ of H , the state $T|\psi\rangle$ is an eigenstate of H that is orthogonal to $|\psi\rangle$.

End